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Higher spin fields in Siegel space, currents and theta functions

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Abstract: Dynamics of four-dimensional massless fields of all spins is formulated in the Siegel space of complex 4×4 symmetric matrices. It is shown that the unfolded equations of free massless fields, that have a form of multidimensional Schrodinger equations, naturally distinguish between positive- and negative-frequency solutions of relativistic field equations, i.e., particles and antiparticles. Multidimensional Riemann theta functions are shown to solve massless field equations in the Siegel space. We establish the correspondence between conserved higher-spin currents in four-dimensional Minkowski space and those in the ten-dimensional matrix space. It is shown that global symmetry parameters of the current in the matrix space should be singular to reproduce a nonzero current in Minkowski space. The $\mathcal{D}-$function integral evolution formulae for 4d massless fields in the Fock-Siegel space are obtained. The generalization of the proposed scheme to higher dimensions and systems of higher ranks is considered.

Keywords: Field Theories in Higher Dimensions, Space-Time Symmetries, Global Symmetries, Differential and Algebraic Geometry

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Contents

1 Introduction

The idea that a set of massless fields of all spins in the four-dimensional space should admit a natural description in the ten-dimensional space $\mathcal{M}_4$ with real symmetric matrix coordi-

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nates $X^{AB} = X^{BA}$ $(A, B, \ldots = 1, \ldots, 4)$ was originally proposed by Fronsdal in [1]. Later, the same conclusion was independently reached in [2]. The dynamical equations in $\mathcal{M}_M$, that for $M = 4$ are equivalent to the field equations for massless fields of all spins in the four-dimensional Minkowski space M^4, are very simple [3]. All integer spin fields of M^4 are described in $\mathcal{M}_4$ by a single scalar field $b(X)$ that satisfies the Klein-Gordon-like equation

$$\Big(\frac{\partial^2}{\partial X^{AB}\partial X^{CD}} - \frac{\partial^2}{\partial X^{AC}\partial X^{BD}}\Big) b(X) = 0\,. \tag{1.1}$$

All half-integer spin fields are described by a single fermion field $f_B(X)$ that satisfies the Dirac-like equation

$$\frac{\partial}{\partial X^{AB}} f_C(X) - \frac{\partial}{\partial X^{AC}} f_B(X) = 0\,. \tag{1.2}$$

The equations (1.1) and (1.2) were derived in [3] from the system of equations

$$\left(\frac{\partial}{\partial X^{AB}} + \mu \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(Y|X) = 0\,, \tag{1.3}$$

where Y^A were treated as auxiliary commuting variables (the parameter $\mu \neq 0$ is introduced for the future convenience). Although the equations (1.1)–(1.3) were originally considered for $M = 4$, they make sense for any M.

The equations (1.3) express the first derivatives with respect to space-time variables X^{AB} in terms of the fields themselves. As such, they belong to the class of *unfolded* partial differential equations (PDE) that, more generally, express the exterior differential of a set of differential forms in terms of exterior products of the differential forms themselves. Such a first-order form of dynamical field equations can always be achieved by introducing a (may be infinite) set of auxiliary fields which parameterize all combinations of derivatives of the dynamical fields that remain non-zero on the field equations. For example, in the system (1.3), the *dynamical fields* are

$$b(X) = C(0|X) \tag{1.4}$$

and

$$f_A(X) = \frac{\partial}{\partial Y^A} C(Y|X)\Big|_{Y=0}\,. \tag{1.5}$$

As a consequence of (1.3), they satisfy, respectively, the equations (1.1) and (1.2). All the fields

$$C_{A_1\ldots A_n}(X) = \frac{\partial^n}{\partial Y^{A_1}\ldots\partial Y^{A_n}} C(Y|X)\Big|_{Y=0}\,, \qquad n > 1 \tag{1.6}$$

are auxiliary, being expressed via higher X-derivatives of the dynamical fields by virtue of the equations (1.3). In [3] it was shown that the equations (1.1) and (1.2) along with constraints that express the auxiliary fields via X-derivatives of the dynamical fields exhaust the content of the unfolded system (1.3). That the equations (1.3) formulated in the ten-dimensional space-time, still describe massless fields in four dimensions was also shown in [3] using the unfolded dynamics approach (see also section 2).

Theories in $\mathcal{M}_M$ have been studied in a number of papers from different perspectives [4–19]. In this paper, we will further study the higher-spin (HS) theory in the matrix space.

The main practical goal is to show how HS conserved currents in $\mathcal{M}_4$ found in [5] reproduce usual HS conserved currents in Minkowski space found in [20]. The analysis is not completely trivial since the conserved charges in $\mathcal{M}_4$ contain an additional integration over one spinning variable. The apparent difficulty is that the compact spin space is contractible to zero, hence implying that the charge must vanish for regular solutions of the field equations in $\mathcal{M}_4$. A standard way out would be to integrate over a noncontractible cycle in $\mathcal{M}_M$. (In fact, in its Sp$(2M)$ invariant compactification which is Lagrangian Grassmannian [1, 4].) We have not been able to proceed along these lines, however. Instead we will show in this paper that the Minkowski charge is correctly reproduced by virtue of introducing a singularity that effectively makes the integration cycle over the spinning variable noncontractible. The obtained results may have several applications.

First of all, conserved currents determine the lowest order Noether interactions with the HS gauge fields associated with the HS symmetries. It is straightforward to introduce cubic interactions of HS gauge potentials with conserved currents via replacing the global symmetry parameters η in the charge by the corresponding HS one-form gauge connections. The results of this paper show that, to reproduce correctly the HS interactions in the four-dimensional setup, HS potentials in $\mathcal{M}_4$ should develop a singularity in the spinning directions. In other words, the results of this paper indicate that there are nontrivial fluxes in the spinning directions in $\mathcal{M}_4$, that support charges in Minkowski space.

Another application is that the obtained formula for conserved charges allows us to write the integral evolution representation for solutions of massless field equations in $\mathcal{M}_4$ with the help of $\mathcal{D}$-functions introduced in [4]. Generically, $\mathcal{D}$-functions provide the integral representation for solution of field equations of the form

$$C(X) = \int_\Sigma \mathcal{D}(X, X'\,)C(X'\,)dX'\ , \tag{1.7}$$

where Σ is a surface where the initial data are given. Formulae of this type should respect a number of properties. Firstly, $\mathcal{D}(X, X'\,)$ should form a solution of the field equations under consideration with respect to X. Secondly, $\mathcal{D}(X, X')\Big|_{X,X'\in\Sigma} = \delta_\Sigma(X - X')$. Thirdly, the formula (1.7) should be independent of local variations of Σ which property is satisfied if $\mathcal{D}(X, X'\,)$ solves the field equations with respect to X' and (1.7) is defined as a conserved charge with respect to X'. The proper definition of the integration measure dX' in (1.7) is achieved in this paper. Note that so defined $\mathcal{D}$-function satisfies the composition property

$$\mathcal{D}(X, X'\,) = \int_\Sigma \mathcal{D}(X, X''\,)\mathcal{D}(X''\,, X'\,)dX''\ .$$

In the analysis of HS currents, we find it most convenient to depart from the real space $\mathcal{M}_M$ to its complexification $\mathfrak{H}_M$ known as upper Siegel half-space [21] (see also [22]). It turns out that in this framework positive- and negative-frequency solutions identify with holomorphic and antiholomorphic solutions in the upper Siegel half-space. $\mathcal{M}_M$ is a boundary (absolute) of the Siegel half-space. A surprising conclusion will be that unfolded field equations themselves distinguish between positive- and negative-frequency solutions of the field equations, i.e., between particles and antiparticles, the property usually delegated to the quantization procedure. It is worth to mention that the unfolded equations have a form of multidimensional Schrodinger equations. These conclusions may eventually be of key importance for a deeper understanding of the interplay between unfolding and quantization.

Once the HS field equations are reformulated in the Siegel space, it is straightforward to observe that they are solved by Riemann theta functions. This fact is truly remarkable in view of the role that theta functions play in modern geometry, theory of integrable systems and String Theory, and is hoped to shed more light on fundamental structures underlying HS theory. Let us note that in some sense theta functions provide most symmetric non-zero solutions of massless field equations. Namely, they are invariant up to a phase under the transformations from the Igusa group $\Gamma_{1,2} \subset \mathrm{Sp}(2M, \mathbb{Z})$ (see [22] for more detail). This class of solutions may indeed play a distinguished role in the HS theory because the observables constructed from such solutions, like, e.g., conserved currents, turn out to be invariant under $\Gamma_{1,2}$. Note also that some of the properties of theta functions admit natural interpretation in terms of the unfolded massless field equations.

All seemingly different aspects of HS theory considered in this paper take their origin in the symmetry properties of the massless field equations (1.3), i.e., HS symmetries. One of the advantages of the unfolded formulation is just that it makes symmetries of PDE manifest. Therefore we start in section 2 by recalling some relevant facts of the unfolded formulation approach, with the emphasize on symmetries in Subsection 2.1. In Subsections 2.2 and 2.3 we recall, respectively, the relationship between the ten-dimensional space $\mathcal{M}_4$ and Minkowski space M^4 and some known results on HS conserved currents in $\mathcal{M}_4$ and M^4. In section 3, we extend the unfolded description of massless higher spins to the complex Siegel space and explain how unfolded field equations distinguish between positive- and negative-frequency solutions. In section 4, we develop further the construction of conserved currents by extending $\mathcal{M}_M$ to $\mathcal{M}_M \times \mathbb{R}^M \times \mathbb{R}^M$ where we introduce a $2M$-form, that is closed by virtue of certain unfolded equations, and show how it reproduces the previously known closed M-forms associated with conserved currents. Then, in section, 5 we show how the known $4d$ currents result from those in $\mathcal{M}_4$ after introducing a singular flux in the spinning directions. Using $\mathcal{D}$-functions found in [4] and the obtained construction of conserved currents, we derive in section 6 the integral evolution formulae for solutions of the massless field equations in $\mathcal{M}_4$. Multilinear conserved currents are considered in section 7. In section 8, we show that Riemann theta functions form a natural class of periodic solutions of massless field equations. Conclusions and perspectives are discussed in section 9. In appendix, we describe a commutative associative product law $\circ$ which endows the space of solutions of unfolded equations of any rank in $\mathcal{M}_M$ with the commutative ring structure.

2 Preliminaries

A natural approach to the study of dynamical equations of motion in the HS gauge theory, referred to as *unfolded formulation*, consists of reformulation of PDE in the form of certain covariant constancy conditions [23]. Using this approach, consistent gauge invariant nonlinear HS equations of motion were found in [24–26] for HS theories in three, four and any dimension, respectively, (see [27–29] for reviews and more references). The unfolded formulation is particularly useful for revealing symmetries and dynamical content of PDE as discussed e.g. in [10]. Here we briefly recall some properties of this approach.

2.1 Unfolded formulation and symmetries

Consider a system of linear PDE of the form

$$(d+\omega)C(X)=0\,,\qquad d=dX^k\frac{\partial}{\partial X^k}\,,\tag{2.1}$$

where $C(X)$ is a section of the trivial vector bundle $\mathcal{B}=\mathbb{R}^d\times V$ over the space-time base $\mathbb{R}^d$ with the local coordinates X^k

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{B} \\ & & \downarrow \\ & & \mathbb{R}^d\,, \end{array}$$

with a linear space V as the fiber. In the cases of interest V identifies with an appropriate space of power series $f(Y)=\sum_{n=0}^{\infty} f_{A_1\dots A_n}(X)Y^{A_1}\dots Y^{A_n}$ in some auxiliary variables Y^A, i.e., $C(X)$ with values in V is realized as a function $C(Y|X)$ of the two types of variables.

The one-form $\omega(X)=dX^k\omega_k(X)$, that satisfies the flatness condition

$$d\omega+\frac{1}{2}[\omega\,,\omega]=0\,,\tag{2.2}$$

is some flat connection of a Lie algebra $\mathfrak{g}\subset End\,V$ with the Lie product $[\,,]$. (Here we discard the wedge product symbol $\wedge$.)

The equation (2.1) is invariant under the global symmetry $\mathfrak{g}$. Indeed, the system (2.1) and (2.2) is invariant under the infinitesimal gauge transformations

$$\begin{aligned}\delta\omega(X)&=d\epsilon(X)+[\omega(X),\epsilon(X)]\,,\\ \delta C(X)&=-\epsilon(X)C(X)\,,\end{aligned}\tag{2.3}$$

where $\epsilon(X)$ is an arbitrary symmetry parameter that takes values in $\mathfrak{g}$. For a given $\omega(X)$, there is a leftover symmetry with the parameter $\epsilon(X)$, that satisfies

$$\delta\omega(X)\equiv d\epsilon(X)+[\omega(X),\epsilon(X)]=0\,.\tag{2.4}$$

This equation on $\epsilon(X)$ is consistent as a consequence of (2.2), i.e., the Bianchi identity $d^2=0$ does not impose any further conditions on $\epsilon(X)$. Therefore, it reconstructs locally the dependence of $\epsilon(X)$ on X in terms of its values $\epsilon(X_0)$ at any point X_0 of space-time. In the absence of topological obstructions, the resulting global symmetry algebra with the parameters $\epsilon(X_0)$ is $\mathfrak{g}$. It is therefore enough to observe that some dynamical system can be reformulated in the form (2.1), where a flat connection $\omega(X)$ takes values in some algebra $\mathfrak{g}$ that acts in V, to reveal the global symmetry $\mathfrak{g}$ (2.3) of the system (2.1). This approach is general since every $\mathfrak{g}$-invariant linear system of PDE can be reformulated in the form (2.1) by adding enough auxiliary variables (nonlinear systems are described in terms of an appropriate generalization associated with free differential algebras [30, 31] as explained e.g. in [9, 19, 23, 27]).

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From this analysis it follows [3] that the equations (1.3) and, hence, (1.1) and (1.2) are invariant under $\mathfrak{sp}(2M,\mathbb{R})$ and its infinite-dimensional HS extension. Indeed, the 0-form $C(Y|X)$ can be interpreted as a section of the fiber bundle $\mathcal{B} = \mathbb{R}^{\frac{M(M+1)}{2}} \times V$ where V is the Fock module of the associative Weyl algebra A_M with the generators Y^A and P_A, that satisfy

$$[P_A\,, Y^B] = \delta^A_B\,, \qquad [Y^A\,, Y^B] = 0\,, \qquad [P_A\,, P_B] = 0\,.$$

The Weyl algebra is spanned by various polynomials $a(Y,P)$. The Fock module V is spanned by the vectors

$$V: \qquad f(Y)|0\rangle$$

generated from the Fock vacuum $|0\rangle$, that satisfies $P_A|0\rangle = 0$. Clearly, V forms a module of A_M as well as of the Lie (super)algebra $(\mathfrak{s})\mathfrak{hs}(2M)$ constructed from A_M via (anti) commutators with even and odd subspaces identified with the spaces of even and odd functions $a(Y,P)$, i.e., $a(-Y,-P) = (-1)^{\pi(a)}a(Y,P)$. The algebras of this type were identified in [32] with the HS symmetry algebras found in [33].[1] Note that we do note rule out the Lie algebra $\mathfrak{hs}$ resulting from A_4 via commutators for all its elements because, although, having wrong relationship between spin and statistics, it is of interest in the context of the theory of theta functions considered in section 8. Note also that bosonic spinorial symmetries of this class were recently considered in the four-dimensional setup in [16, 36, 37].

The finite-dimensional subalgebra $\mathfrak{sp}(2M\,,\mathbb{R}) \subset (\mathfrak{s})\mathfrak{hs}(2M)$ is spanned by the generators

$$L_A{}^B = \frac{1}{2}\{P_A\,, Y^B\}\,, \qquad P_{AB} = P_A P_B\,, \qquad K^{AB} = Y^A Y^B\,.$$

The equations (1.3) now take the form (2.1) with

$$d = dX^{AB}\frac{\partial}{\partial X^{AB}}\,, \qquad \omega = \mu\, dX^{AB} P_{AB}\,. \tag{2.5}$$

Since $[P_{AB}\,, P_{CD}] = 0$, so defined ω is a flat $\mathfrak{sp}(2M)$ connection, that satisfies (2.2). Locally, any flat connection admits a pure gauge representation

$$\omega(P,Y|X) = g^{-1}(P,Y|X)\, d\, g(P,Y|X)\,. \tag{2.6}$$

Using this representation, we solve (2.4) in the form

$$\epsilon(P,Y|X) = g^{-1}(P,Y|X)\epsilon_0(P,Y)g(P,Y|X)\,, \tag{2.7}$$

where $\epsilon_0(P,Y)$ is an arbitrary X-independent element of A_M

$$\epsilon_0(P,Y) = \epsilon \sum_{n,m\geq 0} \eta_{A_1\ldots A_n}{}^{B_1\ldots B_m} Y^{A_1}\ldots Y^{A_n} P_{B_1}\ldots P_{B_m}\,. \tag{2.8}$$

[1] Note that later on it was shown [34] that, although the even subalgebra of the Weyl algebra is indeed the HS symmetry algebra of a nonlinear bosonic HS model with integer spin fields, in presence of fermions both the symmetry and the field spectrum have to be doubled by adding the Klein operators [35] (see [19] for a recent discussion in the context of the analysis in $\mathcal{M}_M$). In this paper we do not introduce the Klein operators, which however should be expected in a nonlinear supersymmetric theory.

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For the flat connection (2.5), the pure gauge representation (2.6) holds with

$$g(P,Y|X) = \exp\left(\mu X^{AB} P_A P_B\right), \qquad g^{-1}(P,Y|X) = \exp\left(-\mu X^{AB} P_A P_B\right).$$

Let $\epsilon_0(P,Y)$ be of the form

$$\epsilon_0(P,Y) = -\epsilon \exp h^B P_B \ \exp j_A Y^A = -\epsilon \exp j_A h^A \ \exp j_A Y^A \ \exp h^B P_B\,,$$

where j_A and h^A are numerical parameters. Then the global symmetry parameter (2.7) takes the form

$$\epsilon(P,Y|X) = -\epsilon \exp j_A h^A \exp\left(-\mu X^{AB} j_A j_B\right) \exp j_A Y^A \ \exp((h^B - 2\mu X^{BC} j_C) P_B)\,.$$

The resulting global symmetry transformation (2.3) reads as

$$\delta C(Y|X) = \epsilon \exp j_A h^A \exp\left(j_A Y^A - \mu X^{AB} j_A j_B\right) C(Y^B + h^B - 2\mu X^{BC} j_C|X)\,. \tag{2.9}$$

Note that this formula was derived in section 7.2 of [3] in the Weyl ordering, i.e., for $\mu = 1$ and

$$\epsilon_0'(P,Y) = \exp(j_A Y^A + h^B P_B) = \exp h^B P_B \exp j_A Y^A \exp\left(-\frac{1}{2} j_A h^B\right).$$

Differentiating (2.9) with respect to h^B and j_A, it is easy to derive the transformation law for any global HS symmetry with polynomial parameter $\epsilon_0(P,Y)$. In particular, at $\mu = 1$ and $M = 4$, the $\mathfrak{sp}(8)$ transformations are [3]

$$\begin{aligned}
P_{AB} C(Y|X) &= \frac{\partial^2}{\partial h^A \partial h^B} \epsilon^{-1} \delta C(Y|X)\Big|_{h^A = j_A = 0} = -\frac{\partial}{\partial X^{AB}} C(Y|X)\,,\\
L_A{}^B C(Y|X) &= \Big(\frac{\partial^2}{\partial h^A \partial j_B} + \frac{M}{2}\delta_A{}^B\Big) \epsilon^{-1} \delta C(Y|X)\Big|_{h^A = j_A = 0}\\
&= \Big(Y^B \frac{\partial}{\partial Y^A} + 2X^{BC} \frac{\partial}{\partial X^{CA}} + \frac{M}{2}\delta_A{}^B\Big) C(Y|X),\\
K^{AB} C(Y|X) &= \frac{\partial^2}{\partial j_A \partial j_B} \epsilon^{-1} \delta C(Y|X)\Big|_{h^A = j_A = 0}\\
= \Big(Y^B Y^A - 2Y^A X^{BC} &\frac{\partial}{\partial Y^C} - 2Y^B X^{AC} \frac{\partial}{\partial Y^C} - 2X^{AB} - 4X^{BC} X^{AD} \frac{\partial}{\partial X^{CD}}\Big)\, C(Y|X)\,.
\end{aligned}$$

Note that, as explained in more detail in [3, 19], $\mathfrak{sp}(8)$ contains $4d$ conformal algebra $\mathfrak{su}(2,2)$ as a subalgebra. Therefore the equations (1.3) are conformal invariant. Irreducible $\mathfrak{su}(2,2)$ invariant subsystems correspond to $4d$ massless fields of different spins.

2.2 Initial data problem

Usual $d-$dimensional Minkowski space-time M^d is a subspace of the matrix space $\mathcal{M}_M$ for an appropriate M. To describe the embedding of $4d$ Minkowski space-time into $\mathcal{M}_4$ it is convenient to use complex notations with two-component indices α, β and α', β' in place of the four-component indices $A, B \ldots$ with the convention that the complex conjugation interchanges unprimed indices $\alpha, \beta = 1, 2$ with the primed ones $\alpha', \beta' = 1', 2'$. We use notation with four-component indices being equivalent to a pair of primed and unprimed two-component Greek indices (e.g., $A = \alpha, \alpha'$) and $Y^A = (Y^\alpha, Y^{\alpha'})$, where $Y^{\alpha'} = \overline{Y^\alpha}$.

In terms of two-component complex spinors we set

$$X^{AB} = \left(X^{\alpha\beta}, X^{\alpha\beta'}, X^{\alpha'\beta'} \right)$$

so that $X^{\alpha'\beta'}$ is complex conjugated to $X^{\alpha\beta}$, i.e., $\overline{X^{\alpha\beta}} = X^{\alpha'\beta'}$, while $X^{\alpha\beta'}$ is hermitian, $\overline{X^{\alpha\beta'}} = X^{\beta\alpha'}$. For Minkowski coordinates, we will sometimes use notation $x^{\alpha\beta'}$ instead of $X^{\alpha\beta'}$. Two-component indices are raised and lowered according to

$$A^\alpha = \varepsilon^{\alpha\beta} A_\beta \,, \qquad A_\beta = \varepsilon_{\alpha\beta} A^\alpha \,, \qquad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \,, \qquad \varepsilon_{12} = 1 \,,$$

and analogously for primed indices.

Minkowski time t and space coordinates x^i are

$$X^{\alpha\beta'} = t\mathcal{T}^{\alpha\beta'} + x^i \sigma_i^{\alpha\beta'} \,, \qquad i = 1, 2, 3 \,, \tag{2.10}$$

where $\mathcal{T}^{\alpha\beta'} = \delta^{\alpha\beta'}$ while $\sigma_i^{\alpha\beta'}$ are hermitian traceless Pauli matrices. The Klein-Gordon and Dirac equations in Minkowski space read as

$$\Big(\frac{\partial^2}{\partial X^{\alpha\beta'} \partial X^{\gamma\delta'}} - \frac{\partial^2}{\partial X^{\alpha\delta'} \partial X^{\gamma\beta'}} \Big) b(X) = 0 \,, \tag{2.11}$$

$$\frac{\partial}{\partial X^{\alpha\beta'}} f_\delta(X) - \frac{\partial}{\partial X^{\delta\beta'}} f_\alpha(X) = 0 \,, \qquad \frac{\partial}{\partial X^{\alpha\beta'}} f_{\gamma'}(X) - \frac{\partial}{\partial X^{\alpha\gamma'}} f_{\beta'}(X) = 0 \,. \tag{2.12}$$

As shown in [4, 9], the generalized space-time $\mathcal{M}_M$ admits the well-defined notions of future and past. The (past)future cones of the origin $X = 0$ are formed by (negative)positive-definite matrices X^{AB}. There is a single time evolution parameter

$$t = \frac{1}{M} X^{AB} \mathcal{T}_{AB} \,, \tag{2.13}$$

where $\mathcal{T}^{AB}$ is some positive-definite time-arrow matrix. For a chosen time parameter t, the global space-like Cauchy surface Σ_t is parameterized as

$$X^{AB} \in \Sigma_t : \qquad X^{AB} = x^{AB} + t\mathcal{T}^{AB} \,, \tag{2.14}$$

where the space coordinates x^{AB} are arbitrary $\mathcal{T}-$ traceless matrices

$$x^{AB} \mathcal{T}_{AB} = 0 \,, \qquad \mathcal{T}_{AB} \mathcal{T}^{BC} = \delta_A^C \,. \tag{2.15}$$

A particular solution in $\mathcal{M}_M$ can be reconstructed from the values of the fields along with some their time derivatives on the global Cauchy surface. However, because the system of equations (1.1) on the field $b(X)$ is overdetermined, some of these equations play a role of constraints that restrict the choice of the initial data on the global Cauchy surface. Independent initial data can be given on a lower-dimensional object called local Cauchy bundle E, which is a $M-$dimensional fiber bundle over a $(d-1)$-dimensional base manifold $\sigma \in \Sigma$ treated as the space manifold. The Minkowski space-time is $R \times \sigma \subset \mathcal{M}_M$ where R is a time axis.

To see that initial data for the equations (1.1), (1.2) should be given on a M-dimensional surface is most convenient by using their unfolded form (1.3). Indeed, the generic solution of (1.3) can locally be given in the form

$$C(Y|X) = \exp\left(-\mu X^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(Y|0)\,,$$

where the "initial data" $C(Y|0)$ is an arbitrary function of M variables Y^A.

Note that the equivalence of the unfolded equations (1.3) to (1.1), (1.2) shown in [3] manifests itself in the inverse formula that reconstructs the dependence of $C(Y|X)$ on Y in terms of any functions $b(X)$ and $f_A(X)$ that satisfy (1.1) and (1.2), respectively,

$$C(Y|X) = \cos(s) b(X) + \frac{\sin(s)}{s} Y^A f_A(X)\,, \qquad s = \sqrt{\frac{1}{\mu} Y^A Y^B \frac{\partial}{\partial X^{AB}}}\,. \tag{2.16}$$

2.3 Higher-spin currents

The infinite set of conformal HS symmetries found in [3] is parameterized by various global symmetry parameters (2.8). This suggests the existence of the corresponding conserved HS currents. Indeed, in [5] it was shown that the M-form in $\mathcal{M}_M$

$$\begin{aligned}\Omega_M(\eta, C^k, C^l) = {} & \epsilon_{C_1\ldots C_M} dX^{C_1 A_1} \wedge \ldots \wedge dX^{C_M A_M} \\ & \eta_{B_1\ldots B_n}{}^{A_{M+1}\ldots A_{M+m}} X^{A_{M+m+1}B_1} \ldots X^{A_{M+m+n}B_n} T^{kl}_{A_1\ldots A_{M+m+n}}(X)\,,\end{aligned} \tag{2.17}$$

where $\epsilon_{C_1\ldots C_M}$ is the totally antisymmetric multispinor and constants η are the HS symmetry parameters, is closed provided that the generalized stress tensor $T^{kl}_{A_1\ldots A_N}(X)$

$$T^{k\,l}{}_{A_1\ldots A_N}(X) = \frac{\partial}{\partial Y^{A_1}} \cdots \frac{\partial}{\partial Y^{A_N}} C^k(Y|X) C^l(iY|X)\Big|_{Y=0} \tag{2.18}$$

is built of the fields $C^k(Y|X)$ and $C^l(iY|X)$ that satisfy (1.3). (Here k and l are color indices which take an arbitrary number of values.) The charge

$$Q(\eta, C^k, C^l) = \int_{E^M} \Omega_M(\eta, C^k, C^l)\,, \tag{2.19}$$

is independent of local variations of a M-dimensional surface E^M, i.e., it conserves.

On the other hand, HS charges in Minkowski space have the form

$$Q(\eta, C^k, C^l) = \int_{\sigma^{d-1}} \Omega_{d-1}(\eta, C^k, C^l)\,, \tag{2.20}$$

where $\Omega_{d-1}(\eta, C^k, C^l)$ is a on-mass-shell closed $(d-1)$-form dual to the conserved current, and σ^{d-1} is a $(d-1)$-dimensional surface in the Minkowski space-time, usually identified with the space surface R^{d-1}. The explicit expression for the on-shell closed three-form $\Omega_3(\eta, C^k, C^l)$ in $4d$ Minkowski space, obtained recently in [20], is

$$\Omega_3(\eta, C^k, C^l) = dx_{\alpha\alpha'} \wedge dx^{\alpha\gamma'} \wedge dx^{\gamma\alpha'} w_\gamma w_{\gamma'} \eta(w,u) C^k(Y|x) C^l(iY|x)\Big|_{Y=0}\,,$$

where

$$w_\alpha = \frac{\partial}{\partial Y^\alpha}, \quad w_{\alpha'} = \frac{\partial}{\partial Y^{\alpha'}}, \quad u^\alpha = x^{\alpha\alpha'}\frac{\partial}{\partial Y^{\alpha'}}, \quad u^{\alpha'} = x^{\alpha\alpha'}\frac{\partial}{\partial Y^\alpha}.$$

Equivalently,[2]

$$\Omega_3(\eta, C^k, C^l) = dx_{\eta\eta'} \wedge dx^{\eta\gamma'} \wedge dx^{\gamma\eta'} \\ \eta_{\alpha(n)\alpha'(m)}{}^{\beta(p)\beta'(q)} x^{\alpha_1\mu'}{}_1 \dots x^{\alpha_n\mu'}{}_n x^{\mu_1\alpha'}{}_1 \dots x^{\mu_m\alpha'}{}_m T^{kl}_{\gamma\beta(p)\mu(m)\gamma'\beta'(q)\mu'(n)}(x) \tag{2.21}$$

is closed provided that $T^{kl}_{A_1\dots A_n}(x)$ is the restriction of the stress tensor (2.18) to the Minkowski space $M^4 \subset \mathcal{M}_4$, that has only nonzero coordinates $X^{\alpha\beta'} = x^{\alpha\beta'}$ among X^{AB}, and the fields $C^k(Y|x)$ satisfy the $4d$ unfolded equations

$$\frac{\partial}{\partial x^{\alpha\beta'}} C^k(Y|x) + \frac{\partial^2}{\partial Y^\alpha \partial Y^{\beta'}} C^k(Y|x) = 0\,. \tag{2.22}$$

Let us note that the equations of this type naturally appear in the study of so-called twistorial world-like particle models (see e.g. [38, 39]) as the Fourier transformation of the Dirac constraints on the respective phase space momenta.

It was conjectured in [5] that the charge (2.19) at $M = 4$ should reproduce the Minkowski HS charge via an appropriate reduction of $\mathcal{M}_4$ to M^4. This conjecture sounds plausible because the two charges contain the same symmetry parameters and the Minkowski stress tensor results from the restriction of the generalized stress tensor in $\mathcal{M}_4$ to the Minkowski space. One of the goals of this paper is to establish the precise correspondence between the $\mathcal{M}_4$ and Minkowski realizations of the conserved charges.

For the case of $M = 2$ the identification is a sort of trivial because $M^3 = \mathcal{M}_2$ and the local Cauchy bundles are the same, both being two-dimensional. (Note that since $3d$ conformal Minkowski charges are constructed from $3d$ massless scalar and spinor they coincide with the $3d$ conformal HS currents found in [40].) For higher dimensions the precise identification is less trivial. The problem is that the dimensions of local Cauchy bundles are different in M^d and $\mathcal{M}_M$ for $M > 2$. The extra dimensions in E^M compared to σ^{d-1} are responsible for spin and are associated with the corresponding compact spaces. It turns out [2, 4, 13] that $E^M = \mathbb{R}^{d-1} \times S^{M-d+1}$ for $M = 2, 4, 8, 16$ correspond to $d = 3,\ 4,\ 6,\ 10$. In particular, the $M = 4$ spin space is S^1.

3 Quantization and Siegel space

The coefficient μ in front of the second term in the unfolded equations (1.3) was irrelevant within an expansion in powers of $Y^{\mathcal{A}}$ used in [3]. Its absolute value can certainly be normalized arbitrarily by a rescaling of Y^A. However, its phase should respect reality conditions. Surprisingly, it distinguishes between positive and negative frequencies, i.e., particles and antiparticles. We interpret this observation as an indication that the unfolded dynamics encodes quantum physics.

[2] We use the convention where a number in parentheses next to an index denotes a number of symmetrized indices. For instance, $\alpha(n)$ stands for n symmetrized indices $\alpha_1 \dots \alpha_n$

Indeed, general solution of the equations (1.1) and (1.2) is [4]

$$\begin{aligned} b(X) &= b^+(X) + b^-(X) \\ &= \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi \Big(b^+(\xi) \exp\big\{ i\xi_A\xi_B X^{AB}\big\} + b^-(\xi) \exp\big\{ - i\xi_A\xi_B X^{AB}\big\}\Big) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} f_A(X) &= f_A{}^+(X) + f_A{}^-(X) \\ &= \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi \xi_A \Big(f^+(\xi) \exp\big\{ i\xi_A\xi_B X^{AB}\big\} + f^-(\xi) \exp\big\{ - i\xi_A\xi_B X^{AB}\big\}\Big). \end{aligned} \tag{3.2}$$

Note that $b^{\pm}(\lambda) = (-1)^M b^{\pm}(-\lambda)$ and $f^{\pm}(\lambda) = (-1)^{M+1} f^{\pm}(-\lambda)$. In particular, for even M, $b^{\pm}(\xi)$ and $f^{\pm}(\xi)$ are even and odd functions of ξ, respectively. The integration in (3.1) and (3.2) is hence over $\mathbb{R}^M/Z_2$. The point $\xi_\alpha = 0$ is invariant under the Z_2 reflection $\xi_\alpha \to -\xi_\alpha$ and therefore is a singular point of the orbifold $\mathbb{R}^M/Z_2$.

Now we observe that the unfolded equations

$$\left(\frac{\partial}{\partial X^{AB}} \pm i\hbar \frac{\partial^2}{\partial Y^A \partial Y^B} \right) C^{\pm}(Y|X) = 0 \tag{3.3}$$

distinguish between the positive- and negative-frequency solutions

$$C^{\pm}(Y|X) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\, c^{\pm}(\xi) \exp \pm i \Big(\hbar\, \xi_A\xi_B X^{AB} + Y^B\xi_B \Big)\,, \tag{3.4}$$

that are complex conjugated to each other for real X and Y,

$$c^-(\xi) = \overline{c^+(\xi)}\,, \qquad C^-(Y|X) = \overline{C^+(Y|X)}\,. \tag{3.5}$$

Note also that, for even M, $c^{\pm}(\xi)$ contain $b^{\pm}(\xi)$ and $f^{\pm}(\xi)$ as even and odd parts, respectively,

$$c^{\pm}(\xi) = b^{\pm}(\xi) + f^{\pm}(\xi)\,. \tag{3.6}$$

As explained in more detail in [4], the manifest decomposition into positive- and negative-frequency parts gives rise to the quantum fields with the creation and annihilation operators $\hat{c}^{\pm}(\xi)$, that satisfy the commutation relations

$$[\hat{c}^{\pm}(\xi_1), \hat{c}^{\pm}(\xi_2)] = 0\,, \qquad [\hat{c}^{-}(\xi_1), \hat{c}^{+}(\xi_2)] = \delta(\xi_1 - \xi_2)\,. \tag{3.7}$$

In this paper we will be mostly interested in the classical picture, however.

For the further analysis it is convenient to introduce complex coordinates

$$\mathcal{Z}^{AB} = X^{AB} + i\,\mathbf{X}^{AB} \;\equiv \mathrm{Re}\,\mathcal{Z}^{AB} + i\,\mathrm{Im}\,\mathcal{Z}^{AB}. \tag{3.8}$$

The real part $\mathrm{Re}\,\mathcal{Z}^{AB}$ of $\mathcal{Z}^{AB}$ is identified with the coordinates of the generalized space-time X^{AB} that contain in particular Minkowski coordinates. The imaginary part $\mathbf{X}^{AB} = \mathrm{Im}\,\mathcal{Z}^{AB}$ is required to be positive definite and was treated in [4] as a regulator that makes the Gaussian integrals well-defined (i.e., physical quantities are obtained in the limit $\mathbf{X}^{AB} \to 0$; note, that the complex coordinates Z^{AB} introduced in [4] are related to $\mathcal{Z}^{AB}$ as $\mathcal{Z}^{AB} = i\overline{Z}^{AB}$). The space of coordinates $\mathcal{Z}^{AB}$ forms the upper Siegel half-space $\mathfrak{H}_M$ [21, 22]. Evidently, $-\overline{\mathcal{Z}}^{AB} \in \mathfrak{H}_M$ provided that $\mathcal{Z}^{AB} \in \mathfrak{H}_M$ and vice versa.

The variables Y^A can also be complexified

$$\mathcal{Y}^A = Y^A + i\,\mathbf{Y}^A,$$

extending the Siegel space to *Fock-Siegel space* $\mathfrak{H}_M \times \mathbb{C}^M$.

The continuations of the functions $C^{\pm}$(3.4) to the Fock-Siegel space $\mathfrak{H}_M \times \mathbb{C}^M$ are

$$C^+(\mathcal{Y}|\mathcal{Z}) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\; c^+(\xi) \exp\left(i\,\hbar\xi_A\xi_B\mathcal{Z}^{AB}\right) \exp i\left(\xi_A\mathcal{Y}^A\right), \tag{3.9}$$

$$C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\; c^-(\xi) \exp\left(-\,i\,\hbar\xi_A\xi_B\overline{\mathcal{Z}}^{AB}\right) \exp -i\left(\xi_A\overline{\mathcal{Y}}^A\right), \tag{3.10}$$

where $c^{\pm}(\xi)$ are some "Fourier coefficients". Depending on a problem in question, they can be chosen to belong to different functional classes.

The broadest framework is provided by distributions that grow not faster than exponentially of order two and zero type at infinity, i.e., not faster than $\exp A|\xi|^2$, $\forall A > 0$. In this case, $c^{\pm}(\xi)$ belong to the space $S'_{1/2}(\mathbb{R}^M)$ dual to the Gelfand-Shilov space $S_{1/2}(\mathbb{R}^M)$.[3] It can be shown,[4] that $C^+(\mathcal{Y}|\mathcal{Z})$ (3.9) and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ (3.10) are, respectively, analytic and antianalytic in $\mathcal{Y}$. The (anti)analyticity of $\left(C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})\right)$ $C^+(\mathcal{Y}|\mathcal{Z})$ in $\mathcal{Z} \in \mathfrak{H}_M$ follows from the integral representations $\big($(3.10)$\big)$ (3.9).

As a subclass, one can require $c^{\pm}(\xi)$ to be infinitely differentiable functions, that grow not faster than exponentially of order two and zero type at infinity. Then, for any $\mathcal{Z} \in \mathfrak{H}_M$ the functions $c^+(\xi)\exp\left(i\,\hbar\xi_A\xi_B\mathcal{Z}^{AB}\right)$ and $c^-(\xi)\exp\left(-\,i\,\hbar\xi_A\xi_B\overline{\mathcal{Z}}^{AB}\right)$ belong to $S_{1/2}(\mathbb{R}^M)$ with respect of ξ and hence their Fourier images $C^+(\mathcal{Y}|\mathcal{Z})$ (3.9) and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ (3.10) belong to $S^{1/2}(\mathbb{R}^M)$. It is worth to note that, as shown in [42, 43], the class $S^{1/2}(\mathbb{R}^M)$ plays a distinguished role in the analysis of convergency of power series in the Moyal star-product in noncommutative field theory. This is particularly interesting taking into account that the interactions of HS fields (for more detail on the role of star-product in HS theories see [3, 25, 26] and reviews [27–29]) is governed by the Moyal star-product, which however acts on the noncommutative spinor variables Y^A rather than on the space-time coordinates as in noncommutative field theory.

[3]Recall that $S_{\alpha_1,\dots,\alpha_M}$ is defined in [41] as a space of infinitely differentiable functions $\phi(x_1,\dots,x_M)$ such that the inequality $\left|x_1^{k_1}\dots x_M^{k_M}\frac{\partial^{q_1+\dots+q_M}}{\partial x_1^{q_1}\dots\partial x_M^{q_M}}\phi(x)\right| \le C_q A_1^{k_1}\dots A_M^{k_M}\,k_1^{k_1\alpha_1}\dots k_M^{k_M\alpha_M}$ holds for any integer nonnegative k_i, q_i and some constants C_q and A_i that depend on ϕ. The Fourier dual space $S^{\alpha_1,\dots,\alpha_M}$ consists of infinitely differentiable functions $\psi(p_1,\dots,p_M)$ that satisfy the inequality $\left|p_1^{k_1}\dots p_M^{k_M}\frac{\partial^{q_1+\dots+q_M}}{\partial p_1^{q_1}\dots\partial p_M^{q_M}}\psi(p)\right| \le C_k B_1^{q_1}\dots B_M^{q_M}\,q_1^{q_1\alpha_1}\dots q_M^{q_M\alpha_M}$ for any integer nonnegative k_i, q_i and some constants C_k and B_i that depend on ψ. $S_{1/2}(\mathbb{R}^M)$ and $S^{1/2}(\mathbb{R}^M)$ are shorthand notations for $S_{\underbrace{1/2,\dots,1/2}_{\mathrm{M}}}$ and $S^{\overbrace{1/2,\dots,1/2}^{\mathrm{M}}}$, respectively.

[4]We are grateful to M.A. Soloviev for communicating to us this fact.

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Further restrictions may be imposed in the case where the fields have to be normalizable with respect to one or another norm. This is needed to guarantee that the bilinear currents are well-defined. As discussed in more detail in Subsection 4.3, the relevant classes of functions $c^{\pm}(\xi)$ include Sobolev spaces $L_2^q(\mathbb{R}^M)$ and Schwartz space $S(\mathbb{R}^M)$.

It is easy to see, that $C^+(\mathcal{Y}|\mathcal{Z})$ and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ are complex conjugated as a consequence of (3.5), i.e., $\overline{C^+(\mathcal{Y}|\mathcal{Z})} = C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ and

$$\left(\frac{\partial}{\partial \mathcal{Z}^{AB}} + i\hbar \frac{\partial^2}{\partial \mathcal{Y}^A \partial \mathcal{Y}^B}\right) C^+(\mathcal{Y}|\mathcal{Z}) = 0\,, \tag{3.11}$$

$$\left(\frac{\partial}{\partial \overline{\mathcal{Z}}^{AB}} - i\hbar \frac{\partial^2}{\partial \overline{\mathcal{Y}}^A \partial \overline{\mathcal{Y}}^B}\right) C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = 0\,. \tag{3.12}$$

The equations (3.11) and (3.12) uplift the massless field equations for (negative)positive frequencies to the Fock-Siegel space. The (anti)holomorphy properties of $C^{\pm}$ reconstruct them in the Fock-Siegel space in terms of theirs boundary values $C^{\pm}(Y|X)$ at $\mathcal{M}_M \times \mathbb{R}^M$. Remarkably, depending on the sign of the second term, the classical unfolded field equations (3.11) and (3.12) distinguish between positive and negative frequencies, the property usually delegated to a quantization prescription. Let us note that this phenomenon also takes place in Minkowski setup with (appropriately complexified) unfolded equations (2.22) as well as in the related twistorial world-line particle models [38, 39]. In this case, the analogues of the upper and lower Siegel spaces are the forward and backward tubes.

The fields $C^{\pm}$ can be unified into the field

$$C(\mathcal{Y}, \overline{\mathcal{Y}}|\mathcal{Z}, \overline{\mathcal{Z}}) = C^+(\mathcal{Y}|\mathcal{Z}) + C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})\,, \tag{3.13}$$

that, however, does not possesses definite (anti)holomorphy properties in $\mathcal{Z}^{AB}$.

4 Bilinear currents

4.1 Current equations

Let us introduce a conserved current which generalizes that of [5] in a way convenient for the further analysis. The key fact is that a differential $2M$-form

$$\varpi^{2M}(g) \;=\; \Big(d\,W_A \wedge \Big(\hbar\, W_B d\,\mathcal{Z}^{AB} - d\,\mathcal{Y}^A\Big)\Big)^M \; g(W,\,\mathcal{Y}\,|\mathcal{Z}) \tag{4.1}$$

is closed in a domain in $\mathbb{C}^{\frac{M(M+1)}{2}}(\mathcal{Z}^{AB}) \times \mathbb{R}^M(W_B) \times \mathbb{C}^M(\mathcal{Y}^A)$ provided that $g(W,\,\mathcal{Y}\,|\mathcal{Z})$ is holomorphic in the variables $\mathcal{Y}$ and $\mathcal{Z}$ and satisfies the following *current* equations

$$\left(\frac{\partial}{\partial \mathcal{Z}^{AB}} + \hbar\, W_{(A} \frac{\partial}{\partial \mathcal{Y}^{B)}}\right) g(W,\,\mathcal{Y}\,|\mathcal{Z}) = 0\,. \tag{4.2}$$

Indeed, from (4.1) and (4.2) it follows that

$$\left(d\,W_A \frac{\partial}{\partial W_A} + d\,\mathcal{Z}^{AB} \frac{\partial}{\partial \mathcal{Z}^{AB}} + d\,\mathcal{Y}^A \frac{\partial}{\partial \mathcal{Y}^A}\right) \wedge \varpi^{2M}(g(W,\,\mathcal{Y}\,|\mathcal{Z})) =$$
$$= \left(d\,W_A \frac{\partial}{\partial W_A} - \Big(\hbar\, W_B d\,\mathcal{Z}^{AB} - d\,\mathcal{Y}^A\Big) \frac{\partial}{\partial \mathcal{Y}^A}\right) \wedge \varpi^{2M}(g(W,\,\mathcal{Y}\,|\mathcal{Z})) = 0$$

because

$$dW_C \wedge \Big(d\,W_A \wedge \Big(\hbar\, W_B d\,\mathcal{Z}^{AB} - d\,\mathcal{Y}^A \Big) \Big)^M = 0$$

and

$$\Big(\hbar\, W_D d\,\mathcal{Z}^{CD} - d\,\mathcal{Y}^C \Big) \wedge \Big(d\,W_A \wedge \Big(\hbar\, W_B d\,\mathcal{Z}^{AB} - d\,\mathcal{Y}^A \Big) \Big)^M = 0\,.$$

As a result, on solutions of (4.2), the charge

$$Q = Q(g) = \int_{\Sigma^{2M}} \varpi^{2M}(g) \tag{4.3}$$

is independent of local variations of a $2M$-dimensional integration surface Σ^{2M}. In particular, for functions that decrease fast enough at space infinity, it is independent of the time parameter in $\mathcal{M}_M$, thus being conserved.

Since (4.2) is a first-order PDE system, the space of its regular solutions forms a commutative algebra $\mathcal{R}$, i.e., a linear combination of products of any regular solutions of (4.2) is also a solution. The algebra $\mathcal{R}$ is formed by functions η of the form

$$\eta(W,\,\mathcal{Y}\,|\mathcal{Z}) = \varepsilon(W_A,\,\mathcal{Y}^C \, - \hbar\,\mathcal{Z}^{CB}\,W_B) \tag{4.4}$$

with arbitrary regular $\varepsilon(W,\mathcal{Y})$. An extension of this property to the space of singular solutions $\mathcal{S}$ is that $\mathcal{S}$ forms an $\mathcal{R}$-module, i.e., although it may not be possible to multiply singular solutions with themselves, they can by multiplied by regular ones.

To make contact with the currents (2.17) note, that eq. (4.1) gives rise to conserved currents for $g(W,\,\mathcal{Y}\,|\mathcal{Z})$ of the form

$$g(W,\,\mathcal{Y}\,|\mathcal{Z}) = \eta(W,\,\mathcal{Y}\,|\mathcal{Z}) f(W,\,\mathcal{Y}\,|\mathcal{Z})\,, \tag{4.5}$$

where $\eta(W,\,\mathcal{Y}\,|\mathcal{Z})$ (4.4) is a polynomial solution of (4.2) identified with a HS symmetry parameter and

$$f(W,\mathcal{Y},|\mathcal{Z}) \;=\; (2\pi)^{-M/2} \int\limits_{\mathbb{R}^M} d^M U \;\; \exp\left(-i\;\, W_C\, U^C\right)\, T(U,\,\mathcal{Y}\,|\mathcal{Z}) \tag{4.6}$$

is a solution built of massless fields via the generalized stress tensor

$$T(U,\,\mathcal{Y}\,|\mathcal{Z}) = C^+(\mathcal{Y} - U|\mathcal{Z})\, C^-(U + \mathcal{Y}|\mathcal{Z}), \tag{4.7}$$

where $C^+(\mathcal{Y}|\mathcal{Z})$ satisfies (3.11) while $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ satisfies (3.12). (For more accurate definition that respects necessary analyticity properties see Subsection 4.2. The appropriate classes of functions $C^+(\mathcal{Y}|\mathcal{Z})$ and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ will be specified in Subsection 4.3.) The inverse transform is

$$T(U,\,\mathcal{Y}\,|\mathcal{Z}) = (2\pi)^{-M/2} \int\limits_{\mathbb{R}^M} d^M\, W \;\; \exp\left(i\;\, W_C\, U^C\right)\, f(W,\;\mathcal{Y}\,|\mathcal{Z})\,. \tag{4.8}$$

The equations (4.2) for $f(W,\ \mathcal{Y}|\mathcal{Z})$ translate to the following equations for the stress tensor

$$\left\{\frac{\partial}{\partial \mathcal{Z}^{AB}} - i\hbar \frac{\partial}{\partial \mathcal{Y}^{(A}} \frac{\partial}{\partial U^{B)}}\right\} T(U,\ \mathcal{Y}|\mathcal{Z}) = 0, \tag{4.9}$$

i.e., $f(W,\ \mathcal{Y}|\mathcal{Z})$ satisfies (4.2) provided that $T(U,\ \mathcal{Y}|\mathcal{Z})$ satisfies (4.9) and vice versa. One can make sure, that the bilinear tensor (4.7) satisfies (4.9) by virtue of (3.11), (3.12).

Note that, up to a factor of $i\hbar$, the equations (4.9) at real $X = \operatorname{Re}\mathcal{Z}$, $Y = \operatorname{Re}\mathcal{Y}$

$$\left\{\frac{\partial}{\partial X^{AB}} - i\hbar \frac{\partial}{\partial Y^{(A}} \frac{\partial}{\partial U^{B)}}\right\} T(U, Y|X) = 0, \tag{4.10}$$

coincide with the rank-2 unfolded equations of [10]. In particular, (4.10) implies

$$\left(\frac{\partial^3}{\partial U^A \partial U^B \partial X^{CD}} + \frac{\partial^3}{\partial U^C \partial U^D \partial X^{AB}} - \frac{\partial^3}{\partial U^C \partial U^B \partial X^{AD}} - \frac{\partial^3}{\partial U^A \partial U^D \partial X^{CB}}\right) T(U, Y|X) = 0\,, \tag{4.11}$$

which equation was used in [5] to prove that the form (2.17) is closed.

4.2 Siegel strip and bilinear currents

To define the integration around singularities in section 5 via a deformation of the integration surface over X^{AB} in $\mathcal{M}_4$ to the complex space, we now introduce a generalized stress tensor that depends on positive- and negative-frequency solutions $C^+(\mathcal{Y}|\mathcal{Z})$ and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ of the unfolded equations (3.11) and (3.12), respectively, and possesses proper holomorphy properties in $\mathfrak{S}_M^{(\mathcal{H})}(\mathcal{Z}) \times \mathbb{C}^M(\mathcal{Y})$, where a domain $\mathfrak{S}_M^{(\mathcal{H})} \subset \mathfrak{H}_M$ will be specified below. In the rest of this section we set $\hbar = 1$.

For a positive definite symmetric matrix $\mathcal{H}^{AB}$ we introduce Siegel $\mathcal{H}$-strip $\mathfrak{S}_M^{(\mathcal{H})} \subset \mathfrak{H}_M$ as follows :

$$\mathcal{Z}^{AB} \in \mathfrak{S}_M^{(\mathcal{H})} \ : \ \begin{cases} (\mathcal{H} - \operatorname{Im}\mathcal{Z})^{AB} & \text{is positive definite} \\ \operatorname{Im}\mathcal{Z}^{AB} & \text{is positive definite} \end{cases}$$

which is a generalization of a strip $0 < \operatorname{Im} z < \mathcal{H}$ in $\mathbb{C}$ to the Siegel space $\mathfrak{H}_M$.

Since $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ is anti-holomorphic in $\mathfrak{S}_M^{(\mathcal{H})} \times \mathbb{C}^M$, one can see that

$$C^-_{\mathcal{H}}(\mathcal{Y}|\mathcal{Z}) = C^-(\mathcal{Y}|\mathcal{Z} - i\mathcal{H}) \tag{4.12}$$

is holomorphic in $\mathfrak{S}_M^{(\mathcal{H})} \times \mathbb{C}^M$. Note, that $\mathcal{H}$ is a parameter of $C^-_{\mathcal{H}}$ and

$$\lim_{\mathcal{H}\to 0} C^-_{\mathcal{H}}(Y|X) = C^-(Y|X) \qquad \forall\ X \in \mathcal{M}_M\,,\ \ Y \in \mathbb{R}^M.$$

It is easy to see that a generalized stress tensor

$$T_{\mathcal{H}}(\mathcal{U},\ \mathcal{Y}|\mathcal{Z}) = C^+(\mathcal{Y} - \mathcal{U}|\mathcal{Z})\, C^-_{\mathcal{H}}(\mathcal{Y} + \mathcal{U}|\mathcal{Z}) \tag{4.13}$$

solves (4.9) and is holomorphic in $\mathfrak{S}_M^{(\mathcal{H})} \times \mathbb{C}^{2M}$. Up to a linear change of variables, its restriction to the real subspace in the limit $\mathcal{H} \to 0$ gives the generalized stress tensor of [5]. To reproduce the charge (2.19) we proceed as follows. Let the integration surface be

$$\boldsymbol{\Sigma}^{2M}_{\mathcal{H}} = \sigma^M_{\mathcal{H}}(X) \times \mathbb{R}^M(W)\Big|_{\{\mathcal{Y} = \mathcal{Y}_0\}}\,,$$

where $\sigma^M_{\mathcal{H}}(X)$ is any $M-$dimensional surface that belongs to the real subspace of $\mathfrak{S}^{(\mathcal{H})}_M$ defined by the equation $\{\,\mathrm{Im}\,\mathcal{Z}=\nu\mathcal{H}\}$, where $0<\nu<1$ is a free parameter. Substituting the $2M$-form $\varpi^{2M}(g)$ (4.1) with the function g (4.5) into (4.3), we obtain

$$Q_{\mathcal{H}}\sim\int\limits_{\boldsymbol{\Sigma}^{2M}_{\mathcal{H}}}\left(dW_A\wedge W_B dX^{AB}\right)^M\,\eta(W,\,\mathcal{Y}_0\,|X+i\nu\mathcal{H})f(W,\,\mathcal{Y}_0\,|X+i\nu\mathcal{H})\,, \tag{4.14}$$

whence, using the Fourier transform (4.6), we get in the limit $\mathcal{H}\to 0$

$$\begin{aligned}Q_0&\sim\int\limits_{\boldsymbol{\Sigma}^{2M}_0}\left(dW_A\wedge W_B dX^{AB}\right)^M\int\limits_{\mathbb{R}^M}d^MU\exp\left(-iW_CU^C\right)\eta(W,\mathcal{Y}_0|X)T_0(U,\mathcal{Y}_0|X) \qquad (4.15)\\&\sim\int\limits_{\sigma^M_0}\epsilon_{A_1\ldots A_M}dX^{A_1B_1}\wedge\ldots\wedge dX^{A_MB_M}\eta\left(-i\frac{\partial}{\partial U},\mathcal{Y}_0|X\right)\frac{\partial}{\partial U^{B_1}}\cdots\frac{\partial}{\partial U^{B_M}}T(U,\mathcal{Y}_0|X)\Big|_{U=0}.\end{aligned}$$

For the parameter η of the form (4.4) with polynomial ε and $\mathcal{Y}_0=0$ this gives (2.19).

Alternatively, one can integrate over $d^M\mathcal{Y}$ at fixed $\mathcal{Z}$. For example let $\boldsymbol{\Sigma}^{2M}=\mathbb{R}^{2M}$: $\{\mathcal{Z}=X_0,\ \mathcal{Y}=Y\}$ for some real X_0. We have in the limit $\mathcal{H}\to 0$

$$\begin{aligned}Q_0&\sim\int\limits_{\mathbb{R}^{2M}}\left(dW_A\wedge dY^A\right)^M\int\limits_{\mathbb{R}^M}d^MU\exp\left(-iW_CU^C\right)\varepsilon\left(W,Y^C-X_0{}^{CB}W_B\right)T_0(U,Y|X_0)\\&\sim\int\limits_{\mathbb{R}^M}\epsilon_{A_1\ldots A_M}dY^{A_1}\wedge\cdots\wedge dY^{A_M}\varepsilon\left(-i\frac{\partial}{\partial U},Y^C+iX_0{}^{CB}\frac{\partial}{\partial U^B}\right)T_0(U,Y|X_0)\Big|_{U=0}. \qquad (4.16)\end{aligned}$$

For C^+ and C^- of the form (3.9) and (3.10), respectively, we have

$$\begin{aligned}&T_0(U,\,Y\,|\,X_0)=\\&\int d^M\xi\,d^M\lambda\,c^+(\xi)c^-(\lambda)\exp i\Big(X_0^{AB}(\xi_A\xi_B-\lambda_A\lambda_B)+Y^A(\xi_A-\lambda_A)-U^C(\lambda_C+\xi_C)\Big).\end{aligned}$$

So we obtain from (4.16)

$$\begin{aligned}Q_0&\sim\int d^M\xi d^M\lambda c^+(\xi)c^-(\lambda)\varepsilon\left(-\lambda-\xi,\frac{1}{2i}\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\lambda}\right)\right)\delta^M(\xi-\lambda) \qquad (4.17)\\&\sim\int d^M\xi\left(\varepsilon\left(-2\xi,\frac{i}{2}\frac{\partial}{\partial\nu}\right)c^+(\xi+\nu)c^-(\xi-\nu)\right)\Big|_{\nu=0}\sim\int d^M\xi c^+(\xi)\varepsilon\left(-2\xi,\frac{i}{2}\frac{\partial}{\partial\xi}\right)c^-(\xi).\end{aligned}$$

As expected, the result is independent of X_0. Up to a rescaling of arguments it reproduces the alternative expression for the charge obtained in [5].

The important improvement of the form of the current (4.3) compared to (2.17) is that it allows us to introduce singularities (fluxes) in the spinning variables W_α, $W_{\alpha'}$, $\mathcal{Y}^\alpha,\mathcal{Y}^{\alpha'}$, $\mathcal{Z}^{\alpha\beta},\mathcal{Z}^{\alpha'\beta'}$ that can bring in a proper singularity into the charge (4.16) which, as we show in section 5, is needed to reproduce HS currents of [20] in Minkowski space-time.

Finally, let us note that the generalized stress tensor can also be constructed from arbitrary frequency fields as follows. Let fields $\mathcal{C}_j(\mathcal{Y}|\mathcal{Z})$ of frequencies $\eta_j=\pm1$ satisfy the equations

$$\left(\frac{\partial}{\partial\mathcal{Z}^{AB}}+i\,\eta_j\,\frac{\partial^2}{\partial\mathcal{Y}^A\partial\mathcal{Y}^B}\right)\mathcal{C}_j(\mathcal{Y}|\mathcal{Z})=0\,. \tag{4.18}$$

Then the generalized stress tensor

$$T(\mathcal{U},\mathcal{Y}|\mathcal{Z})=\mathcal{C}_j\Big(\alpha_j(\mathcal{Y}-\mathcal{U}))|\mathcal{Z}\Big)\;\mathcal{C}_k\Big(\beta_k(\mathcal{U}+\mathcal{Y}))|\mathcal{Z}\Big)\qquad\Big((\alpha_j)^2=\eta_j,\;(\beta_k)^2=-\eta_k\Big)$$

satisfies (4.9). Note however that the charge (4.3) built of regular parameters (4.4) and fields of equal frequencies, that are holomorphic in the same Fock-Siegel space, vanishes because an integration surface can be deformed to the respective infinity in the imaginary coordinates $\mathcal{Z}^{AB}$ where the fields $\mathcal{C}_j(\mathcal{Y}|\mathcal{Z})$ vanish. For the fields of opposite frequencies such a deformation is not possible, that results in a non-vanishing charge (may be, after an appropriate $\mathcal{H}-$shift).

4.3 Appropriate classes of solutions

The formula (4.17) for the charge was obtained under assumption that the integrals under consideration are well-defined. The following four options are most significant

(i) $c^{\pm}(\xi)\in L_2(\mathbb{R}^M)$,

(ii) $c^{\pm}(\xi)\in L_2^q(\mathbb{R}^M)$,

(iii) $c^{\pm}(\xi)\in S(\mathbb{R}^M)$,

(iv) $c^{\pm}(\xi)\in S_{1/2}(\mathbb{R}^M)$ and $c^{\mp}(\xi)\in S'_{1/2}(\mathbb{R}^M)$.

The cases (i)–(iii) are appropriate,[5] respectively, for the cases of $\varepsilon=1$ (electric charge), degree q polynomial $\varepsilon\left(\frac{\partial}{\partial\xi}\right)$ and generic polynomial $\varepsilon\left(-2\xi,\frac{i}{2}\frac{\partial}{\partial\xi}\right)$. The case (iv) is appropriate to pair a generalized function with a test one. This is relevant to the analysis of the composition formulae for $\mathcal{D}$-functions in Subsection 6.4. As mentioned in section 3, in all these cases $C^+(\mathcal{Y}|\mathcal{Z})$ (3.9) is analytic and $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ (3.10) is anti-analytic in $\mathcal{Y}$ and $\mathcal{Z}$ provided that $\mathcal{Z}\in\mathfrak{S}_M^{(\mathcal{H})}(\mathcal{Z})$.

Then the generalized bilinear stress tensor (4.13) is

$$\begin{aligned}T_{\mathcal{H}}(U,\,Y|\mathcal{Z})&=C^+(Y-U|\mathcal{Z})\,C^-(Y+U|\mathcal{Z}-i\mathcal{H})\\&=\int d^M\lambda d^M\xi c^+(\xi)c^-(\lambda)\exp i\Big(\hbar\xi_A\xi_B\mathcal{Z}^{AB}-\hbar\lambda_A\lambda_B(\mathcal{Z}^{AB}-i\mathcal{H}^{AB})\\&\qquad+\xi_A(Y^A-U_A)-\lambda_B(U^B+Y^B)\Big).\end{aligned}\tag{4.19}$$

Its Fourier transform (4.6) is

$$\begin{aligned}f_{\mathcal{H}}(W,\,Y|\mathcal{Z})\sim&\int d^M\chi c^+\Big(\frac{-W+\chi}{2}\Big)c^-\Big(\frac{-W-\chi}{2}\Big)\\&\times\exp i\Big(-\hbar W_A\chi_B\mathcal{Z}^{AB}+\hbar\frac{i}{4}(W+\chi)_A(W+\chi)_B\mathcal{H}^{AB}+\chi_AY^A\Big).\end{aligned}\tag{4.20}$$

Note, that for all cases (i)-(iv)

$$c^+\Big(\frac{-W+\chi}{2}\Big)c^-\Big(\frac{-W-\chi}{2}\Big)\in S'_{1/2}(\mathbb{R}^{2M})\,.\tag{4.21}$$

[5]Recall that $L_p^q(\mathbb{R}^M)$ is the Sobolev space, and $S(\mathbb{R}^M)$ is the Schwartz space of infinity differentiable functions $f(\xi)$ that decrease at infinity with all their derivatives faster than any multi-degree of $\frac{1}{|\xi_j|}$.

As a result, since $\mathcal{Z} \in \mathfrak{S}_M^{(\mathcal{H})}(\mathcal{Z})$, $f_{\mathcal{H}}(W, Y|\mathcal{Z})$ (4.20) is integrable over $\mathbb{R}^M(W)$ $\forall(\mathcal{Z}, Y) \in \mathfrak{S}_M^{(\mathcal{H})}(\mathcal{Z}) \times \mathbb{R}^M$. Moreover, the exponential factor with $\mathcal{Z} \in \mathfrak{S}_M^{(\mathcal{H})}(\mathcal{Z})$ guarantees that $f_{\mathcal{H}}(W, \mathcal{Y}|\mathcal{Z})$ is analytic in $\mathcal{Y}$ and $\mathcal{Z}$.

As an illustrative example, let us consider functions of the form

$$c^{\pm}(\xi) = P^{\pm}(\xi)\, \exp(-E^{\pm AB}\xi_A\xi_B) \tag{4.22}$$

with some polynomial $P^{\pm}(\xi)$ and positive definite symmetric $E^{\pm AB}$, that allows explicit analysis. Evidently, the integrals (4.17) converge absolutely for any polynomial parameter ε. Evaluating the Gaussian integral, we have

$$\begin{aligned} f_{\mathcal{H}}(\mathcal{W}, \mathcal{Y}|\mathcal{Z}) = {} & (2\pi)^{-M/2} \int d^M U \exp\left(-i\mathcal{W}_C U^C\right) \\ & P^{+}\Big(i\frac{\partial}{\partial U}\Big) \det{}^{-\frac{1}{2}}(-i\mathcal{Z} + E^{+}) \exp\Big(-\frac{1}{4}(-i\mathcal{Z} + E^{+})^{-1}{}_{AB}(U^A - \mathcal{Y}^A)(U^B - \mathcal{Y}^B)\Big) \\ & P^{-}\Big(i\frac{\partial}{\partial U}\Big) \det{}^{-\frac{1}{2}}(\mathcal{H} + i\mathcal{Z} + E^{-}) \exp\Big(-\frac{1}{4}(\mathcal{H} + i\mathcal{Z} + E^{-})^{-1}{}_{AB}(U^A + \mathcal{Y}^A)(U^B + \mathcal{Y}^B)\Big). \end{aligned} \tag{4.23}$$

The real parts $\mathrm{Re}(\mathcal{H} + i\mathcal{Z} + E^{-})$ and $\mathrm{Re}(-i\mathcal{Z} + E^{+})$ are positive definite provided that $\mathcal{Z}$ belongs to the Siegel $\mathcal{H}-$ strip. Since from $\mathcal{F} \in \mathfrak{H}_M$ follows $\mathcal{F}^{-1} \in \mathfrak{H}_M$ (see e.g. [22]), $\mathrm{Re}\big((\mathcal{H} + i\mathcal{Z} + E^{-})^{-1}\big)$ and $\mathrm{Re}\big((-i\mathcal{Z} + E^{+})^{-1}\big)$ are also positive definite. Hence $f_{\mathcal{H}}(\mathcal{W}, \mathcal{Y}|\mathcal{Z})$ (4.23) is holomorphic in its arguments for $\mathcal{Z} \in \mathfrak{S}_M^{(\mathcal{H})}$. Note that the "additional" analyticity of $f_{\mathcal{H}}(\mathcal{W}, \mathcal{Y}|\mathcal{Z})$ in $\mathcal{W}$ takes place for any $c^{\pm}(\xi) \in S_{1/2}^{1/2}(\mathbb{R}^M)$, which is the case for (4.22).

5 From $\mathcal{M}_4$ to Minkowski space

5.1 Idea of construction

In [5], it was conjectured that the charge (2.19) should reproduce the HS charges (2.20) in Minkowski space M^4 [20] by an appropriate reduction to $M^4 \subset \mathcal{M}_4$. Since the charge in $\mathcal{M}_4$ contains four integrations versus three in the Minkowski space, the naive reduction with the fourth integration over a cyclic spin variable in $\mathcal{M}_4$ gives zero because the cycle is contractible. To make the cycle noncontractible, a singularity (flux) should be introduced in the spinning space. As we demonstrate now, this can be done using the generalized current (4.1), which result was hard to achieve starting from the original expression (2.17).

As explained in section 2, the embedding of the $4d$ Minkowski space-time into $\mathcal{M}_4$ is conveniently described in the language of two-component complex spinors. In these terms, $W^A = (W^\alpha, W^{\alpha'})$ and $U^A = (U^\alpha, U^{\alpha'})$. It should be stressed that the complex structures of the Siegel space and of two-component spinors are different. Correspondingly, since both the real and imaginary parts of the complex variables $\mathcal{Z}^{AB}$ are real symmetric matrices, in terms of two-component complex spinors we have

$$\mathcal{Z}^{AB} = (\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha\alpha'}, \mathcal{Z}^{\alpha'\beta'}) = (X^{\alpha\beta} + i\mathbf{X}^{\alpha\beta}, X^{\alpha\alpha'} + i\mathbf{X}^{\alpha\alpha'}, X^{\alpha'\beta'} + i\mathbf{X}^{\alpha'\beta'})$$

with

$$\overline{X^{\alpha\beta}} = X^{\alpha'\beta'}, \quad \overline{X^{\alpha\beta'}} = X^{\beta\alpha'}, \qquad \overline{\mathbf{X}^{\alpha\beta}} = \mathbf{X}^{\alpha'\beta'}, \quad \overline{\mathbf{X}^{\alpha\beta'}} = \mathbf{X}^{\beta\alpha'}.$$

Note that $\mathrm{Re}\, \mathcal{Z}^{\alpha\beta} = \mathrm{Re}\, X^{\alpha\beta} - \mathrm{Im}\, \mathbf{X}^{\alpha\beta}$, $\mathrm{Im}\, \mathcal{Z}^{\alpha\beta} = \mathrm{Im}\, X^{\alpha\beta} + \mathrm{Re}\, \mathbf{X}^{\alpha\beta}$, *etc.* Analogously we introduce $\mathcal{Y}^A = (\mathcal{Y}^\alpha, \mathcal{Y}^{\alpha'})$.

Let us choose the integration surface $\mathbf{\Sigma}^8$ in the form

$$\mathbf{\Sigma}^8 = \sigma^3(\mathcal{Z}^{\alpha\beta'}) \times \sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y} - \mathcal{Y}_0) \times \sigma^4(W)\,, \tag{5.1}$$

where $\sigma^3(\mathcal{Z}^{\alpha\beta'})$ is a three-dimensional surface in the complexified Minkowski space ${}^{\mathbb{C}}M^4$, $\sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y})$ is a one-dimensional cycle in the "spinning" subspace, $\sigma^4(W) \subseteq \mathbb{R}^4(W)$ is a four-dimensional surface and $\mathcal{Y}_0$ is a free parameter.

An elementary calculation then shows that the pullback of the differential form (4.1) to the integration surface $\mathbf{\Sigma^8}$ (5.1) gives

$$\varpi^{2M}(g)\big|_{\mathbf{\Sigma}^8} \sim d\mathcal{Z}_{\alpha\gamma'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\gamma'} \wedge d\Lambda(W, \mathcal{Y}|\mathcal{Z}) \wedge d^4 W \; g(W, \mathcal{Y}|\mathcal{Z}) W_\beta W_{\beta'} \Big|_{\mathbf{\Sigma}^8}, \tag{5.2}$$

where

$$\Lambda(W,\, \mathcal{Y}\,|\mathcal{Z}) = W_\mu W_\nu\, \mathcal{Z}^{\mu\nu} - W_{\mu'} W_{\nu'}\, \mathcal{Z}^{\mu'\nu'} - W_\mu\, \mathcal{Y}^\mu + W_{\mu'}\, \mathcal{Y}^{\mu'}. \tag{5.3}$$

The key observation is that for any $\mathcal{Y}_0$ the function $\Lambda(W, \mathcal{Y} - \mathcal{Y}_0\,|\mathcal{Z})$ solves (4.2) and is independent of $\mathcal{Z}^{\alpha\beta'}$. This allows us to use Λ to introduce a singularity in a way independent of the complexified Minkowski coordinates $\mathcal{Z}^{\alpha\beta'}$.

To obtain Minkowski charge we set

$$g_\Lambda(W,\, \mathcal{Y}\,|\mathcal{Z}) = \Lambda^{-1}(W, \mathcal{Y}|\mathcal{Z})\, g(W,\, \mathcal{Y}\,|\mathcal{Z})\,. \tag{5.4}$$

Since the functions $\Lambda(W, \mathcal{Y}\,|\mathcal{Z})$ and $g(W,\, \mathcal{Y}\,|\mathcal{Z})$ solve the equations (4.2) , the same is true for $g_\Lambda(W,\, \mathcal{Y}\,|\mathcal{Z})$ (5.4) away from singularities. From (5.2) we have

$$\int_{\mathbf{\Sigma^8}} \varpi^{2M}(g_\Lambda) \sim \int_{\mathbf{\Sigma^8}} d\mathcal{Z}_{\alpha\gamma'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\gamma'} \wedge \frac{d\Lambda}{\Lambda} \wedge d^4\, W\; g(W,\, \mathcal{Y}\,|\mathcal{Z})\, W_\beta\, W_{\beta'}\,. \tag{5.5}$$

The idea is to choose one-dimensional cycle such that $\dfrac{d\Lambda}{\Lambda} = i\, d\phi$, where $\phi \in [0, 2\pi)$ to be a real coordinate on σ^1. This can be done as follows

$$\begin{gathered}\sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y}) = \\ \{\mathcal{Z}^{\alpha\beta} = \rho S^{\alpha\beta} \exp(i\phi), \quad \mathcal{Z}^{\alpha'\beta'} = \rho S^{\alpha'\beta'} \exp(i\phi), \quad \mathcal{Y}^\alpha = \rho S^\alpha \exp(i\phi), \quad \mathcal{Y}^{\alpha'} = \rho S^{\alpha'} \exp(i\phi)\},\end{gathered} \tag{5.6}$$

where $\rho > 0$ is a real parameter and at least some of parameters $S^{\alpha\beta}$, $S^{\alpha'\beta'}$, S^α, and $S^{\beta'}$ are non-zero.

For this choice we obtain

$$\Lambda(W,\, \mathcal{Y}\,|\mathcal{Z})\big|_{\sigma^1} = \rho \exp(i\phi) P(W), \tag{5.7}$$

where

$$P(W) = W_\alpha W_\beta S^{\alpha\beta} - W_{\alpha'} W_{\beta'} S^{\alpha'\beta'} - W_\alpha S^\alpha + W_{\alpha'} S^{\alpha'}\,. \tag{5.8}$$

Hence $\frac{d\Lambda}{\Lambda} = i\,d\phi\,,$ and a residue is at $\rho = 0$ for $P(W) \neq 0$.

The subtlety that the integrand of the right side of (5.5) is not defined at $P(W) = 0$ does not affect the result for any $g(W, \mathcal{Y}\,|\mathcal{Z})$ integrable over $\mathbb{R}^4(W)$ because $P(W)$ cancels out in (5.5) and the variety $P:\ P(W) = 0$ has measure zero. As a result, we obtain

$$Q \sim \int_{\mathbf{\Sigma^8}} \varpi^{2M}(g_\Lambda) = 2i\pi \int_{\sigma^3\times\mathbb{R}^4} d^3\,\mathcal{Z}^{\beta\beta'} \wedge d^4W \;\; g(W,\,\mathcal{Y}_0\,|\mathcal{Z}^{\alpha\alpha'})W_\beta\,W_{\beta'}\,, \tag{5.9}$$

where $d^3\mathcal{Z}^{\beta\beta'} = d\mathcal{Z}_{\alpha\alpha'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\alpha'}$.

For

$$g(W,\,\mathcal{Y}\,|\mathcal{Z}) = \,\eta(W,\,\mathcal{Y}\,|\mathcal{Z})f(W,\,\mathcal{Y}\,|\mathcal{Z})\,, \tag{5.10}$$

where $\eta(W,\,\mathcal{Y}\,|\mathcal{Z})$ is a polynomial solution (4.4) of (4.2) while $f(W,\,\mathcal{Y}\,|\mathcal{Z})$ is the Fourier transform (4.6) of the generalized stress tensor $T(U,\,\mathcal{Y}\,|\mathcal{Z})$ we have

$$\begin{aligned} Q \sim & \int_{\sigma^3\times\mathbb{R}^4} d^3\,\mathcal{Z}^{\beta\beta'} \wedge d^4W \;\, \eta(W, \mathcal{Y}_0|\mathcal{Z}^{\alpha\alpha'})f(W,\mathcal{Y}_0|\mathcal{Z})W_\beta W_{\beta'} \\ & \sim \int_{\sigma^3} d^3\,\mathcal{Z}^{\beta\beta'}\frac{\partial^2}{\partial U^\beta\partial U^{\beta'}}\left(\eta\left(i\frac{\partial}{\partial\,U^C},\,\mathcal{Y}_0\,|\mathcal{Z}^{\alpha\alpha'}\right)T(U,\mathcal{Y}_0|\mathcal{Z})\right)\Big|_{U=0}\,, \end{aligned} \tag{5.11}$$

which is just the anticipated expression for conserved charge in Minkowski space.

To give precise meaning to this construction it remains to identify $\sigma^3(\mathcal{Z}^{\alpha\beta'}) \subset {}^{\mathbb{C}}M^4$, to choose appropriate Siegel strip $\mathcal{H}$, replacing the generalized stress tensor $T(U,\,\mathcal{Y}\,|\mathcal{Z})$ by $T_\mathcal{H}(U,\,\mathcal{Y}\,|\mathcal{Z})$ (4.13) bilinear in the massless fields $C^\pm$. This is done in the next Subsection.

5.2 Integration cycle

Let a positive definite matrix $\mathcal{H} \in \mathcal{M}_M$ be chosen in such a way that $\mathcal{H}^{\alpha\beta} = \mathcal{H}^{\alpha'\beta'} = 0$. Let $\sigma^3 = \sigma^3_\mathcal{H}(\mathcal{Z}^{\alpha\beta'})$ be a real three-dimensional surface in a real subspace of ${}^{\mathbb{C}}M^4$ defined by the condition $\operatorname{Im}\mathcal{Z}^{\alpha\beta'} = \nu\mathcal{H}^{\alpha\beta'}$, where $0 < \nu < 1$ is a free parameter. Note, that with this choice, both $\operatorname{Im}\mathcal{Z}^{\alpha\beta'}$ and $\mathcal{H}^{\alpha\beta'} - \operatorname{Im}\mathcal{Z}^{\alpha\beta'}$ are positive definite for $\forall\mathcal{Z}^{\alpha\beta'} \in \sigma^3_\mathcal{H}$.

Let $\sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y})$ be chosen in the form (5.6) such that both

$$\mathcal{N}^{AB} = \Big(\operatorname{Im}\left(\rho {S_0}^{\alpha\beta}\exp(i\phi)\right),\, \nu\mathcal{H}^{\alpha\beta'}\,,\, \operatorname{Im}\left(\rho {S_0}^{\alpha'\beta'}\exp(i\phi)\right)\Big)$$

and

$$\mathcal{H}^{AB} - \mathcal{N}^{AB}$$

are positive definite, which is true for sufficiently small ρ because this is the case for $\rho = 0$.

For this choice we obtain that $\sigma^3_\mathcal{H} \times \sigma^1 \subset \mathfrak{S}^{(\mathcal{H})}_M \times \mathbb{C}^4(\mathcal{Y})$.

Now let $T = T_\mathcal{H}(U,\,\mathcal{Y}\,|\mathcal{Z})$ be the stress tensor (4.13) built of the massless fields $C^\pm$ with the help of a positive definite regulator matrix $\mathcal{H}$. Let $f_\mathcal{H}(W,\,\mathcal{Y}\,|\mathcal{Z})$ in (5.10) be the Fourier transform (4.6) of $T = T_\mathcal{H}(U,\,\mathcal{Y}\,|\mathcal{Z})$. If $C^\pm$ are of the form (3.9), (3.10) with functions $c^\pm$ from one of the classes (i)–(iv) of Subsection 4.3, then the pullback of the function $g(W,\, \mathcal{Y}\,|\mathcal{Z})$ (5.10) to the integration surface $\mathbf{\Sigma}^8$ (5.1) is integrable over $\mathbb{R}^4(W)$ and the considerations of the previous subsection are true. Therefore (5.11) acquires the form

$$Q = Q_{\mathcal{H}} \sim \int\limits_{\sigma^3_{\mathcal{H}}} d^3\, X^{\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \left(\eta \left(i\frac{\partial}{\partial\, U^C},\, \mathcal{Y}_0\,|\mathcal{Z}^{\alpha\alpha'} \right) T_{\mathcal{H}}(U, \mathcal{Y}_0|\mathcal{Z}) \right) \Big|_{U=0} .$$

In the limit $\mathcal{H} \to 0$, this gives for η (4.4)

$$Q_0 \sim \int\limits_{\sigma^3_0} d^3\, X^{\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \left(\varepsilon \left(-i\frac{\partial}{\partial\, U^C},\, \mathcal{Y}_0 +\ i\, X^{AB} \frac{\partial}{\partial\, U^B} \right) T_0(U,\, \mathcal{Y}_0\,|X) \right) \Big|_{U=0} . \tag{5.12}$$

Recall that according to (4.13)

$$T_0(U,\, \mathcal{Y}|X) = C^+(\mathcal{Y} - U|X)\, C^-(\mathcal{Y} + U|X)\, .$$

Up to a rescaling of variables, for $\mathcal{Y}_0 = 0$ this gives the conserved charge in Minkowski space of [20] for polynomial $\varepsilon(W,\, \mathcal{Y})$.

Let us stress that Q (5.12) is $\mathcal{Y}_0$ independent, i.e., $\frac{\partial}{\partial \mathcal{Y}_0^A} Q = 0\,$, because the variation over $\mathcal{Y}_0^A$ is equivalent to a local variation of the integration cycle away from singularities. For $\varepsilon = const$ and T of the form (4.13) this gives the following identity

$$0 = \int\limits_{\sigma^3} d^3 X^{\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \Big(C^+(\mathcal{Y} - U|X) \frac{\partial}{\partial \mathcal{Y}^A} C^-(\mathcal{Y} + U|X) + C^-(\mathcal{Y} + U|X) \frac{\partial}{\partial \mathcal{Y}^A} C^+(\mathcal{Y} - U|X) \Big) \Big|_{U=0} \tag{5.13}$$

which will be used in the further analysis.

6 Integral evolution formulae

6.1 $\mathcal{D}$-functions

$\mathcal{D}-$functions of the massless field equations (1.1) and (1.2) in $\mathcal{M}_M$ were introduced in [4] as their singular solutions resulting from the integral representation (3.4) with $c^\pm = \mp i\pi^{-\frac{M}{2}}$ (in this section we set $\hbar = 1$),

$$\mathcal{D}^+(\mathcal{Z}) = -\frac{i}{\pi^M} \int\limits_{\mathbb{R}^M} d^M \xi\ \exp i(\xi_A \xi_B \mathcal{Z}^{AB}), \tag{6.1}$$

$$\mathcal{D}^-(\overline{\mathcal{Z}}) = -\mathcal{D}^+(-\overline{\mathcal{Z}}) = \overline{\mathcal{D}^+(\mathcal{Z})} \tag{6.2}$$

and

$$\mathcal{D}(\mathcal{Z}, \overline{\mathcal{Z}}) = \mathcal{D}^+(\mathcal{Z}) + \mathcal{D}^-(\overline{\mathcal{Z}}) = \mathcal{D}^+(\mathcal{Z}) - \mathcal{D}^+(-\overline{\mathcal{Z}})\, .$$

By construction, the functions $\mathcal{D}^-(\overline{\mathcal{Z}})$, $\mathcal{D}^+(\mathcal{Z})$ and $\mathcal{D}(\mathcal{Z},\overline{\mathcal{Z}})$ solve the equations of motion (1.1). For $\mathcal{Z}$ in the upper Siegel half-space $\mathfrak{H}_M$, (6.1) gives

$$\mathcal{D}^+(\mathcal{Z}) = -i\pi^{-\frac{M}{2}} s^{-1}\,, \tag{6.3}$$

where

$$s^2 = \det(-i\mathcal{Z}) \tag{6.4}$$

defines a multidimensional hyperelliptic surface and s is chosen to be holomorphic for $\mathcal{Z} = X + i\mathbf{X} \in \mathfrak{H}_M$ (i.e., positive definite $\mathbf{X}^{AB}$) and to be positive real for purely imaginary $\mathcal{Z}$, i.e., $X = 0$. As shown in [4]

$$\mathcal{D}^+(\mathcal{Z})\Big|_{\mathbf{X}\to 0} = -\frac{i}{\pi^{\frac{M}{2}}} \exp\frac{i\pi I_X}{4} \frac{1}{\sqrt{|\det(X)|}}\Big|_{\mathbf{X}\to 0}\,,$$

where I_X is the inertia index of the matrix X^{AB}, i.e., $I_X = n_+ - n_-$, where n_+ and n_- are, respectively, the numbers of positive and negative eigenvalues of X^{AB}.

From (6.2) and (6.3) it follows that

$$\mathcal{D}^-(\overline{\mathcal{Z}}) = i\pi^{-\frac{M}{2}} \bar{s}^{-1}\,,$$

where $\bar{s}$ is complex conjugated to s (6.4).

The dependence of the $\mathcal{D}-$functions on $\mathcal{Y}$, $\overline{\mathcal{Y}}$ is reconstructed via the unfolded equations (3.11) and (3.12), respectively. In particular,

$$\mathcal{D}^+(\mathcal{Y}|\mathcal{Z}) = \frac{-i}{\pi^M}\int d^M\xi\, \exp i(\xi_A\xi_B\mathcal{Z}^{AB} + \mathcal{Y}^A\xi_A), \tag{6.5}$$

leading to

$$\mathcal{D}^+(\mathcal{Y}|\mathcal{Z}) = \frac{-i}{\pi^{M/2}}\, s^{-1} \exp\left(-\frac{i}{4}\mathcal{Z}_{AB}\mathcal{Y}^A\mathcal{Y}^B\right)\,, \qquad \mathcal{Z}_{AB}\mathcal{Z}^{BC} = \delta_A^C\,. \tag{6.6}$$

Notice that $\mathcal{D}^+(\mathcal{Y}|\mathcal{Z})$ (6.6) behaves as $-2^M i\delta^M(\mathcal{Y})$ at $\mathcal{Z}\to 0$.

6.2 Evolution formula in $\mathcal{M}_4$

The obtained results allow us to give precise meaning to the integral formula (1.7) in $\mathcal{M}_4$ by defining the integration measure as corresponding to the electric charge case in Minkowski space. This guarantees that the restriction of so defined integral representation to the Minkowski space correctly reproduces the dynamics of massless fields.

Namely, we apply formula (5.12) with the generalized stress tensor (4.13) at $\mathcal{H}\to 0$, where $C^-(Y|X)$ is the restriction of $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ to the real subspace while $C^+(\mathcal{Y}|\mathcal{Z})$ is replaced by $\mathcal{D}^+(\mathcal{Y}-\overline{\mathcal{Y}}|\mathcal{Z}-\overline{\mathcal{Z}})$ with $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Z}}$ interpreted as parameters. (Recall that if $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathfrak{H}_M$ then $\mathcal{Z}_1 - \overline{\mathcal{Z}_2} \in \mathfrak{H}_M$ as well, including the case, where either $\mathcal{Z}_1$ or $\mathcal{Z}_2$ is real, belonging to the boundary of $\mathfrak{H}_M$.) Let the resulting conserved charge (5.12) with $\varepsilon = \frac{1}{2}$, which is a function of $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Z}}$, be denoted as $\widetilde{C}^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$. We obtain

$$\widetilde{C}^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = \frac{1}{2}\int_{\sigma^3} d^3X'^{\,\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \Big(\mathcal{D}^+(Y_0 - U - \overline{\mathcal{Y}}|X' - \overline{\mathcal{Z}})C^-(Y_0 + U|X')\Big)\Big|_{U=0}, \tag{6.7}$$

where Y_0 is a free parameter, $\sigma^3 \subset M_4(X)$ is an Euclidean three-dimensional subspace $\mathbb{R}^3$ of Minkowski space,

$$\sigma^3(X) = \{\mathcal{T}_{\alpha\beta'} X^{\alpha\beta'} = t_0\}, \tag{6.8}$$

where a positive-definite matrix $\mathcal{T}_{\alpha\beta'}$ describes the time arrow, the time evolution parameter is $t = \frac{1}{2}\mathcal{T}_{\alpha\beta'} X^{\alpha\beta'}$ and t_0 is its value associated with the chosen space surface. For example, for $\mathcal{T}_{\alpha\beta'} = \delta_{\alpha\beta'}$, $X^{\alpha\beta'} = t\delta^{\alpha\beta'} + x^i \sigma_i^{a\beta'}$, where $\sigma_i^{a\beta'}$ are Pauli matrices.

The key observation is that from (6.7) it follows by virtue of (5.13) that all derivatives of $\widetilde{C}^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}})$ with respect to $\overline{\mathcal{Y}}$ are related to those of $C^-(\mathcal{Y}|X')$ with respect to $\mathcal{Y}$ just in the same way as $\widetilde{C}^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}})$ and $C^-(\mathcal{Y}|X')$ in (6.7), i.e.,

$$\frac{\partial}{\partial \overline{\mathcal{Y}}^{A_1}} \cdots \frac{\partial}{\partial \overline{\mathcal{Y}}^{A_m}} \widetilde{C}^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}}) = \tag{6.9}$$

$$\frac{1}{2}\int_{\sigma^3} d^3X'^{\,\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \Big(\mathcal{D}^+(Y_0 - U - \overline{\mathcal{Y}}|X' - \overline{\mathcal{Z}}) \frac{\partial}{\partial U^{A_1}} \cdots \frac{\partial}{\partial U^{A_m}} C^-(Y_0 + U|X')\Big)\Big|_{U=0}.$$

Let us now show that from (6.7) it follows that

$$\widetilde{C}^-(0\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3} = C^-(0\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3}. \tag{6.10}$$

Using that $d^3X^{\gamma\gamma'}\Big|_{\sigma^3} = d^3x\,\mathcal{T}^{\gamma\gamma'}\Big|_{\sigma^3}$, we obtain from (6.7)

$$\widetilde{C}^-(0|\overline{\mathcal{Z}}) = i\frac{1}{2}\int_{\sigma^3} d^3x' \left(C^-(0|X')\,\dot{\mathcal{D}}^+(0|X' - \overline{\mathcal{Z}}) - \dot{C}^-(0|X')\,\mathcal{D}^+(0|X' - \overline{\mathcal{Z}})\right), \tag{6.11}$$

where we have taken into account that the functions $\mathcal{D}^+$ and C^- satisfy the unfolded equations (3.11) and (3.12), respectively, together with the facts that

$$\frac{\partial}{\partial U^A}\mathcal{D}^+(-U|X - \overline{\mathcal{Z}})\Big|_{U=0} = 0$$

as a consequence of (6.6) and

$$\mathcal{T}^{\alpha\alpha'}\frac{\partial^2}{\partial U^\alpha \partial U^{\alpha'}} C^\pm(U|X) = \pm i\dot{C}^\pm(U|X)\,, \qquad \dot{C}^\pm(U|X) = \frac{\partial}{\partial t} C^\pm(U|X)\,. \tag{6.12}$$

To prove (6.10), it is convenient to use the complex notation for the space coordinates x^j [4]

$$x = x^1 + ix^2\,, \qquad \bar{x} = x^1 - ix^2\,,$$

such that $dx_1 \wedge dx_2 \wedge dx_3 = \frac{1}{2i} dx_3 \wedge dx \wedge d\bar{x}$.

The combinations of ξ_α dual to the coordinates x^3, x, $\bar{x}$ [4]

$$k_3 = \xi_1\bar{\xi}_{\dot{1}} - \xi_2\bar{\xi}_{\dot{2}}\,, \qquad \overline{k} = 2\xi_1\bar{\xi}_{\dot{2}}\,, \qquad k = 2\bar{\xi}_{\dot{1}}\xi_2 \tag{6.13}$$

map $\mathbb{R}^4/Z_2$ on $\mathbb{R}^3$, i.e. k_3, k and $\overline{k}$ can take arbitrary values. The leftover ambiguity in the integration variables ξ_α for a given k_i is the overall phase factor $\xi_\alpha \to \exp\frac{1}{2}i\varphi\,\xi_\alpha$, $\varphi \in [0, 2\pi)$. (Recall that ξ_α is identified with $-\xi_\alpha$.) We set

$$\exp i\phi = 2\frac{\xi_1\xi_2}{k}\,, \qquad \exp -i\phi = 2\frac{\overline{\xi}_{\dot{1}}\overline{\xi}_{\dot{2}}}{\overline{k}}\,. \tag{6.14}$$

The integration measures are related as follows [4]

$$dk_3 \wedge dk \wedge d\bar{k} \wedge d\phi = -8i(\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}})d\xi_1 \wedge d\bar{\xi}_{\dot{1}} \wedge d\xi_2 \wedge d\bar{\xi}_{\dot{2}}\,. \tag{6.15}$$

The map (6.13), (6.14) from $\mathbb{R}^4/Z_2$ associated with the variables ξ_α to $\mathbb{R}^3 \times S^1$ described by the variables k_i, ϕ is non-degenerate except for the expected singularity at $\xi_\alpha = 0$. Note that

$$\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}} = \sqrt{k\bar{k} + k_3^2} = \sqrt{k_1^2 + k_2^2 + k_3^2}\,. \tag{6.16}$$

For any C^+ (3.9) and C^- (3.10) we have using formulae (6.13)–(6.16)

$$C^\pm(0|t,\, x) \;=\; \frac{1}{\pi^2}\int d^4\xi\; c^\pm(\xi)\exp \pm i(tk_0 + x^3k_3 + x\bar{k} + \bar{x}k)\,, \tag{6.17}$$

$$\dot{C}^\pm(0|t,\, x) \;=\; \frac{1}{\pi^2}\int d^4\xi\; (\pm\, i\, k_0)\, c^\pm(\xi)\exp \pm i(tk_0 +\, x^3k_3 + x\bar{k} + \bar{x}k)\,. \tag{6.18}$$

Using (6.17) and (6.18), we obtain from (6.11)

$$\begin{aligned}
\widetilde{C}^-(0\,|t_0, x) \;=\;& \frac{i}{4\pi^6}\int\limits_{\sigma^3} dx'_3 \wedge dx' \wedge d\bar{x}' \int dp_3 \wedge dp \wedge d\bar{p} \wedge d\psi\, dk_3 \wedge dk \wedge d\bar{k} \wedge d\phi \\
& \frac{i}{64}\left(\frac{1}{p_0}+\frac{1}{k_0}\right) c^-(p,\psi)\exp -i\left(t_0p_0 + x'^3p_3 + x'\bar{p} + \bar{x}'p\right) \\
& \exp i\left(\left(x'^3 - x^3\right)k_3 + (x' - x)\,\bar{k} + (\bar{x}' - \bar{x})k\right) \\
=\;& \frac{1}{\pi^2}\int dp_3 \wedge dp \wedge d\bar{p} \wedge d\psi \frac{i}{8p_0}\; c^-(p,\psi)\;\; \exp i(-t_0p_0 - x^3p_3 - x\bar{p} - \bar{x}p) \\
=\;& C^-(0|t_0, x)\,.
\end{aligned}$$

Analogously, from (6.9) it follows that

$$\widetilde{C}^-{}_{A_1\ldots A_m}(\,0\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3} = C^-{}_{A_1\ldots A_m}(\,0\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3}\,, \tag{6.19}$$

where $C^-(\overline{\mathcal{Y}}\,|\,\overline{\mathcal{Z}})_{A_1\ldots A_m} \equiv \dfrac{\partial}{\partial\overline{\mathcal{Y}}^{A_1}}\cdots\dfrac{\partial}{\partial\overline{\mathcal{Y}}^{A_m}}C^-(\overline{\mathcal{Y}}\,|\,\overline{\mathcal{Z}})$. This is equivalent to

$$\widetilde{C}^-(\overline{\mathcal{Y}}\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3} = C^-(\overline{\mathcal{Y}}\,|\,\overline{\mathcal{Z}})\Big|_{\bar{\mathcal{Z}}\in\sigma^3}\,.$$

Because the dependence on $\overline{\mathcal{Y}}$ fully reconstructs the dependence on $\overline{\mathcal{Z}}$ by virtue of the unfolded equation (3.12) from here it follows that

$$\widetilde{C}^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}}) = C^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}})\,, \tag{6.20}$$

that proves the evolution formula

$$C^-(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}}) = \frac{1}{2}\int\limits_{\sigma^3} d^3X'^{\,\beta\beta'}\frac{\partial^2}{\partial U^\beta \partial U^{\beta'}}\Big(\mathcal{D}^+(Y_0 - U - \overline{\mathcal{Y}}|X' - \overline{\mathcal{Z}})C^-(Y_0 + U|X'\,)\Big)\Big|_{U=0}. \tag{6.21}$$

Analogously one can see that

$$C^+(\mathcal{Y}\,|\mathcal{Z}) = \frac{1}{2}\int\limits_{\sigma^3} d^3X'^{\,\beta\beta'}\frac{\partial^2}{\partial U^\beta \partial U^{\beta'}}\Big(\mathcal{D}^-(Y_0+U-\mathcal{Y}|X'\,-\mathcal{Z})\,C^+(Y_0-U|X'\,)\Big)\Big|_{U=0}. \quad (6.22)$$

Note, that from here the usual formulae for spin 0 and spin 1/2 fields follow

$$C^-(0|\overline{\mathcal{Z}}) = i\frac{1}{2}\int\limits_{\sigma^3} d^3x'\Big(C^-(0|X'\,)\dot{\mathcal{D}}^+(0|X'-\overline{\mathcal{Z}}) - \dot{C}^-(0|X'\,)\mathcal{D}^+(0|X'-\overline{\mathcal{Z}})\Big),$$

$$C^-_\alpha(0|\overline{\mathcal{Z}}) = \int\limits_{\sigma^3} d^3x'\mathcal{T}^{\beta\beta'}C^-_\beta(0|X'\,)\mathcal{D}^+_{\alpha\beta'}(0|X'-\overline{\mathcal{Z}})\,,$$

where $\mathcal{D}^+_{\alpha\beta'}(\mathcal{Y}|\mathcal{Z}) = \frac{\partial}{\partial\mathcal{Y}^\alpha}\frac{\partial}{\partial\mathcal{Y}^{\beta'}}\mathcal{D}^+(\mathcal{Y}|\mathcal{Z})$, etc.

6.3 Evolution formula in the Fock space

Now let us show that the same result can be obtained for any M from (4.16) at $\varepsilon = \frac{i}{(2i)^M}$ by the integration over the $\mathcal{Y}$ variables as

$$C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = \frac{i}{(2i)^M}\int\limits_{\mathcal{Y}'=Y'} d^M\mathcal{Y}'\;\mathcal{D}^+(\mathcal{Y}'-\overline{\mathcal{Y}}|\mathcal{Z}'-\overline{\mathcal{Z}})C^-(\mathcal{Y}'|\mathcal{Z}') \quad (6.23)$$

with any $\mathcal{Z}'$. Indeed, substituting

$$C^-(\mathcal{Y}|\mathcal{Z}) = \int d^M\xi\; c^-(\xi)\exp -i\big(\xi_A\mathcal{Y}^A + \xi_A\xi_B\mathcal{Z}^{AB}\big)$$

into (6.23) and using the expression (6.5) for $\mathcal{D}^+$, we obtain

$$\begin{aligned}&\frac{i}{(2i)^M}\int d^MY'\;\mathcal{D}^+(Y'-\overline{\mathcal{Y}}|\mathcal{Z}'-\overline{\mathcal{Z}})\,C^-(Y'|\mathcal{Z}')\\ &= \frac{1}{(2\pi i)^M}\int d^MY'd^M\xi d^M\lambda\; c^-(\xi)\\ &\qquad\exp -i\big(\lambda_A(\overline{\mathcal{Y}}-Y')^A + \lambda_A\lambda_B(\overline{\mathcal{Z}}-\mathcal{Z}')^{AB} + \xi_AY'^A + \xi_A\xi_B\mathcal{Z}'^{AB}\big)\\ &= \int d^M\xi\; c^-(\xi)\exp -i\big(\xi_A\overline{\mathcal{Y}}^A + \xi_A\xi_B\overline{\mathcal{Z}}^{AB}\big) \;=\; C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}).\end{aligned}$$

6.4 Composition properties of the $\mathcal{D}$-functions

Let us consider

$$C^+(\mathcal{Z}|\mathcal{Y}) = \mathcal{D}^+(\mathcal{Z}|\mathcal{Y}) \quad \text{and} \quad C^-(\overline{\mathcal{Z}}|\overline{\mathcal{Y}}) = \mathcal{D}^-(\overline{\mathcal{Z}}-\mathcal{Z}'|\overline{\mathcal{Y}}-\mathcal{Y}'), \quad (6.24)$$

where $\mathcal{Z}' \in \mathfrak{H}_M$ and $\mathcal{Y}' \in \mathbb{C}^M$ are free parameters. One can see, that

$$C^-(\overline{\mathcal{Z}}|\overline{\mathcal{Y}}) = \int d^M\xi\, c^-(\xi)\exp -i\left(\xi_A\xi_B\overline{\mathcal{Z}}^{AB} + \overline{\mathcal{Y}}^A\xi_A\right),$$

$$c^-(\xi) = \frac{i}{\pi^M}\exp i\left(\xi_A\xi_B\mathcal{Z}'^{AB} + \mathcal{Y}'^A\xi_A\right)$$

and

$$C^{+}(\mathcal{Z}|\mathcal{Y}) = \int d^M\xi\, c^{+}(\xi) \exp i(\xi_A \xi_B \mathcal{Z}^{AB} + \mathcal{Y}^A \xi_A)\,, \quad c^{+}(\xi) = \frac{-i}{\pi^M}\,.$$

Since $c^{-}(\xi) \in S_{1/2}$ and $c^{+}(\xi) \in S'_{1/2}$ for any $\mathcal{Z}' \in \mathfrak{H}_M$, we can pair C^{+} and C^{-} (6.24) in the evolution formula (6.22) to obtain

$$\mathcal{D}^{-}(\overline{\mathcal{Y}}\,|\overline{\mathcal{Z}}) = \frac{1}{2}\int\limits_{\sigma^3} d^3X'^{\,\beta\beta'} \frac{\partial^2}{\partial U^{\beta}\partial U^{\beta'}} \Big(\mathcal{D}^{+}(-U - \overline{\mathcal{Y}}|X' - \overline{\mathcal{Z}})\mathcal{D}^{-}(U|X')\Big)\Big|_{U=0}. \qquad (6.25)$$

Analogously from (6.21) it follows that

$$\mathcal{D}^{+}(\mathcal{Y}\,|\mathcal{Z}) = \frac{1}{2}\int\limits_{\sigma^3} d^3X'^{\,\beta\beta'} \frac{\partial^2}{\partial U^{\beta}\partial U^{\beta'}} \Big(\mathcal{D}^{-}(U - \mathcal{Y}|X' - \mathcal{Z})\,\mathcal{D}^{+}(-U|X')\Big)\Big|_{U=0}. \qquad (6.26)$$

These formulae express the composition properties of the $\mathcal{D}$-functions.

6.5 Solutions associated with the $\mathcal{D}$-functions

Finally let us note that one can use the formula (6.6) to generate new solutions of the HS equations as follows. Since $\mathcal{D}^{+}(\mathcal{Y}|\mathcal{Z})$ solves the HS field equations for any $\mathcal{Y}$,

$$\frac{1}{\sqrt{\det(i\mathcal{Z})}} \exp\Big(-\frac{i}{4}\mathcal{Z}_{AB}(\mathcal{Y}^A + uV^A)(\mathcal{Y}^B + uV^B)\Big)\,,$$

where V^A is any constant vector and u is a constant parameter, also does. Integrating this expression at $\mathcal{Y} = 0$ with respect to u with some weight $\rho(u)$, we arrive at the following set of solutions of the equation (1.1)

$$C(0|\mathcal{Z}) = f\Big(V^A V^B \mathcal{Z}_{AB}\Big) \det{}^{-\frac{1}{2}}(i\mathcal{Z})\,, \qquad (6.27)$$

where f is an arbitrary double differentiable function. One can check directly that $C(0|X)$ (6.27) solves (1.1).

7 Higher rank conserved currents

In [10], we have introduced higher rank fields in $\mathcal{M}_M$. Positive- and negative-frequency rank r fields satisfy the unfolded equations of the form

$$\left\{\frac{\partial}{\partial X^{AB}} \pm i\,\eta_{k\,l}\,\frac{\partial^2}{\partial Y_k{}^B \partial Y_l{}^A}\right\} C^{\pm}(Y\,|X) = 0\,, \qquad (7.1)$$

where $k, l = 1 \ldots r$ and η_{kl} is a positive definite symmetric form. In [10] it was also argued that rank r fields in $\mathcal{M}_M$ can be interpreted as resulting from the reduction of rank 1 fields in $\mathcal{M}_{rM}$ to $\mathcal{M}_M$ diagonally embedded into $\mathcal{M}_{rM}$ via

$$X^{(A,k)(B,l)} = \eta^{kl} X^{AB}\,. \qquad (7.2)$$

A particular realization of higher rank fields is provided by products of lower rank fields. For example, a product of two rank 1 fields

$$C(Y_1, Y_2\,|X) = C(Y_1|X)C(Y_2|X)$$

gives a rank 2 field.

Analogously, we introduce a higher rank generalization of the current equation (4.2)

$$\left\{ \frac{\partial}{\partial X^{AB}} + W_{k(A}\frac{\partial}{\partial Y_{k}{}^{B)}} \right\} g(W,\,Y\,|X) = 0\,, \tag{7.3}$$

where $g(W,\,Y\,|X)$ takes values in $\mathcal{M}_M \times \mathbb{R}^{r\,M} \times \mathbb{R}^{r\,M}$ with coordinates X^{AB}, $W_{jA}, Y_j{}^A$, $A,\,B = 1, \dots M$, $j = 1, \dots r$. Again, particular solutions of the higher rank current equation are provided by the products of lower rank currents. The higher rank current equation allows us to derive multilinear conserved currents. Indeed one can easily see that the $2rM$ differential form

$$\varpi_{2rM}\Big(g(W,\,Y\,|X)\Big) \;=\; \Big(d\,W_{kA} \wedge \Big(W_{kB} d\,X^{AB} - d\,Y_k{}^A\Big)\Big)^{r\,M} g(W,\,Y\,|X) \tag{7.4}$$

is closed provided that $g(W,\,Y\,|X)$ satisfies (7.3). As a result, on solutions of (7.3) the charge

$$Q^r = \int_{\Sigma^{2r\,M}} \varpi_{2rM}(g) \tag{7.5}$$

is independent of local variations of a $2rM$-dimensional surface Σ^{2rM}. In particular, it is time-independent hence providing a conserved charge.

The formula (7.4) gives rise to conserved currents for $g(W,\,Y\,|X)$ of the form (4.5) where the "symmetry parameter" $\eta(W,\,Y\,|X)$ is of the form

$$\eta(W,\,Y\,|X) = \varepsilon(W_{jA},\,Y_k{}^C \;-\; X^{CB}\,W_{kB}) \tag{7.6}$$

and $f(W,\,Y\,|X)$ is a solution of (7.3) related to a multilinear "stress tensor" $T(U,\,Y\,|X)$ via the Fourier transform

$$f(W,\,Y\,|X) = (2\pi)^{-rM/2}\int d^{r\,M}U \;\; \exp\left(-i\;\,W_{jC}\,U^{jC}\right)\,T(U,\,Y\,|X)\,, \tag{7.7}$$

$$T(U,\,Y\,|X) = (2\pi)^{-rM/2}\int d^{r\,M}\,W \;\; \exp\left(i\;\,W_{jC}\,U^{jC}\right)\,f(W,\;Y\,|X)\,. \tag{7.8}$$

The equations (7.3) are equivalent to the following equations for T

$$\left\{ \frac{\partial}{\partial X^{AB}} - i\,\frac{\partial}{\partial Y_j{}^{(A}}\frac{\partial}{\partial U^{jB)}} \right\} T(U,\,Y|X) = 0, \tag{7.9}$$

i.e., $f(W,\,Y\,|X)$ satisfies (7.3) provided that $T(U,\,Y\,|X)$ satisfies (7.9) and vice versa.

A $2r-$multilinear "stress tensor" $T(U,\,Y\,|X)$ can be constructed analogously to the bilinear one [5] as follows

$$T(U,\,Y\,|X) = \prod_{j=1}^{r} C^{+}{}_j(Y_j - U_j|X)\,C^{-}{}_j(U_j + Y_j|X), \tag{7.10}$$

where $C^{\pm}{}_1(Y|X)$, ..., $C^{\pm}{}_r(Y|X)$ are solutions of positive- or negative-frequency rank 1 equations.

The higher rank form (7.4) can be interpreted as the pullback of the standard form (4.1) in $\mathcal{M}_{rM}$ to the diagonal subspace $\mathcal{M}_M \subset \mathcal{M}_{rM}$ (7.2). The multilinear stress tensor (7.10) is nothing but the bilinear stress tensor (2.18) on the rank one solutions in $\mathcal{M}_{rM}$, that result from rank r solutions in $\mathcal{M}_M$ provided by the r-linear products of the rank 1 solutions in $\mathcal{M}_M$.

The formula (7.10) gives rise to multilinear currents built of free massless fields. Naively, the higher conserved currents constructed from this stress tensor amount to algebraic functions of the bilinear currents. This is indeed true if the integration measure factorizes into a product of lower-rank measures, that concerns both the integration surface and the parameter function $\eta(W,Y|X)$ (7.6), but may not be true e.g. for nonpolynomial $\eta(W,Y|X)$ that contains a singular dependence analogous to (5.4) in the case of Minkowski bilinear current. Note that the resulting nontrivial currents should be nonlocal from the perspective of usual Minkowski space-time because the charge (7.5) contains $2rM$ integrations instead of $2M$ in the rank 1 case. Specific examples of higher-rank currents will be considered elsewhere.

Note that, analogously to the consideration of sections 3 and 4, all higher rank constructions can be complexified. Namely, one can consider holomorphic continuation $C(\mathcal{Y}^A_k|\mathcal{Z}^{BC})$ of the fields $C(Y^A_k|X^{BC})$, that take values in $\mathfrak{H}_M(\mathcal{Z}) \times \mathbb{C}^{rM}(\mathcal{Y})$.

Finally let us note that the higher rank system (7.1) is invariant under the group $O(r) \times \mathbb{R}$ that acts as follows

$$\mathcal{Y}^A_i \to \widetilde{\mathcal{Y}}^A_i = \exp(\phi)\, T_i{}^j \mathcal{Y}^A_j\,, \qquad \mathcal{Z}^{AB} \to \widetilde{\mathcal{Z}}^{AB} = \exp(2\phi)\, \mathcal{Z}^{AB}\,,$$

where $\phi \in \mathbb{R}$ and $T_i{}^j \in O(r)$ leaves invariant the metric tensor η_{jk}. This symmetry can be used for the derivation of identities between solutions of the equations (7.1) using the fact that if two functions $C_1(\mathcal{Y}|\mathcal{Z})$ and $C_2(\mathcal{Y}|\mathcal{Z})$ satisfy (7.1) and coincide at some $\mathcal{Z} = \mathcal{Z}_0$, $C_1(\mathcal{Y}|\mathcal{Z}_0) = C_2(\mathcal{Y}|\mathcal{Z}_0)$, then they coincide for any $\mathcal{Z}$.

For example, the following identities hold for the $\mathcal{D}^{\pm}$-functions

$$\mathcal{D}^{\pm}(\mathcal{Y}_1|\mathcal{Z}) \ldots \mathcal{D}^{\pm}(\mathcal{Y}_r|\mathcal{Z}) = \exp(rM\phi)\, \mathcal{D}^{\pm}(\widetilde{\mathcal{Y}}_1|\widetilde{\mathcal{Z}}) \ldots \mathcal{D}^{\pm}(\widetilde{\mathcal{Y}}_r|\widetilde{\mathcal{Z}})\,. \tag{7.11}$$

In particular, for the rank 2 case we obtain

$$2^M \mathcal{D}^{\pm}(2U|2\mathcal{Z}) \mathcal{D}^{\pm}(2\mathcal{Y}|2\mathcal{Z}) = \mathcal{D}^{\pm}(U+\mathcal{Y}|\mathcal{Z}) \mathcal{D}^{\pm}(U-\mathcal{Y}|\mathcal{Z})\,. \tag{7.12}$$

Identities (7.11) are continuous analogues of the well-known generalized Riemann identities of theta functions.

8 Riemann theta functions as solutions of higher-spin equations

8.1 Theta functions in the Fock-Siegel space

A somewhat surprising property of the massless field equations formulated in the Fock-Siegel space $\mathfrak{H}_M \times \mathbb{C}^M$ is that Riemann theta functions form their natural solutions. Indeed, a general positive-frequency solution (3.9) periodic under $\mathcal{Y}^A \to \mathcal{Y}^A + n^A$, $n^A \in \mathbb{Z}^M$ has the form

$$C^+(\mathcal{Y}|\mathcal{Z}) = \sum_{n^A \in \mathbb{Z}^M} c_n^+ \exp i(\hbar \mathcal{Z}^{AB}(2\pi n_A)(2\pi n_B) + 2\pi n_C \mathcal{Y}^C) . \tag{8.1}$$

With $c_n^+ = 1$ and $\hbar = \frac{1}{4}\pi^{-1}$ this formula gives the standard expression for the Riemann theta function [22]

$$\theta(\mathcal{Y},\, \mathcal{Z}) = \sum_{n^A \in \mathbb{Z}^M} \exp i\pi(\mathcal{Z}^{AB} n_A n_B + 2 n_A \mathcal{Y}^A) . \tag{8.2}$$

Here the complexified space-time coordinates $\mathcal{Z}^{AB}$ identify with the complex period matrix that defines quasi-periods of $\theta(\mathcal{Y},\, \mathcal{Z})$,

$$\theta(\mathcal{Y} + m\mathcal{Z},\, \mathcal{Z}) = \exp\left(-i\pi \mathcal{Z}^{AB} m_A m_B - 2i\pi m_A \mathcal{Y}^A\right) \theta(\mathcal{Y},\, \mathcal{Z}) , \qquad m_A \in \mathbb{Z}^M .$$

Also let us note that theta function is $\mathcal{Z}$ periodic in the sense

$$\theta(\mathcal{Y}\, , \mathcal{Z} + \mathcal{Z}_{\rm int}) = \theta(\mathcal{Y}\, , \mathcal{Z}) ,$$

where $\mathcal{Z}^{AB}_{\rm int}$ is any real symmetric matrix with integer elements and even diagonal elements.

The fundamental reason why theta functions solve the HS field equations is that both HS theory [1, 3] and the theory of theta functions [22] are based on the $\mathrm{Sp}(2M)$ symmetry and its Weyl-Heisenberg extension which, on the HS gauge theory side, is just the HS symmetry. For example, as mentioned in Introduction, for the case of $M = 2$ the $\mathfrak{sp}(4) \sim \mathfrak{o}(3,2)$ identifies with the conformal symmetry of massless scalar and spinor in three space-time dimensions. In the case of $M = 4$, the appearance of the $\mathfrak{sp}(8)$ symmetry, that acts on the infinite sets of all bosonic and all fermionic massless fields, is a less trivial fact observed originally in [1]. The equation (1.3), which is the simplest $\mathfrak{sp}(8)$ invariant unfolded equation, was shown in [3] to describe properly massless fields of all spins in four dimensions (a closely related argument was also given in [2]). On the other hand, it is well-known that theta functions form a $\Gamma_{1,2}$-module, where $\Gamma_{1,2}$ is the Igusa subgroup of $\mathrm{Sp}(2M|\mathbb{Z})$ [22].

Thus, the fact that theta functions satisfy the same equations as HS fields is not too mysterious once $\mathrm{Sp}(2M)$ appeared in the HS theory. Moreover, from the HS theory perspective, the $\Gamma_{1,2}$ symmetry in the theory of theta functions is the leftover symmetry of the continuous HS symmetry that leaves invariant a particular solution of the HS field equations provided by the theta function up to a phase. This class of solutions may indeed play a distinguished role in the HS theory because conserved currents constructed from such solutions, turn out to be invariant under $\Gamma_{1,2}$.

The roles of the variables $\mathcal{Z}^{AB}$ and $\mathcal{Y}^A$ in the HS theory and the theory of theta functions is to some extent opposite. In the HS theory, $\mathcal{Z}^{AB}$ are space-time variables while the twistor variables $\mathcal{Y}^A$ play an auxiliary role at least in the conventional field-theoretical picture. The indices $A = 1, 2 \ldots M$ for $M = 2^k$ are interpreted as spinorial on the HS theory side. In the theory of theta functions, M identifies with genus g, the period matrix $\mathcal{Z}^{AB}$ (usually denoted Ω^{AB} [22]) plays a role of a parameter, while the dependence on $\mathcal{Y}^A$

(usually denoted z^A) is of most interest. Note however, that in the nonlinear HS theory it was realized since nineties (see [27] and references therein) that the fundamental HS dynamics is encoded in terms of the twistor variables $\mathcal{Y}^A$, while the role of $\mathcal{Z}^{AB}$ is to visualize the HS dynamics in terms of local events [4].

Many of the well-known properties of theta functions acquire a nice interpretation in terms of the HS symmetry (2.9) of the fundamental unfolded equation (3.11) which is a multidimensional analogue of the Schrodinger equation. For example, theta functions with characteristics $b^A \in \mathbb{R}^M$, $a_A \in \mathbb{R}^M$

$$\begin{aligned}\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\mathcal{Y},\,\mathcal{Z}) &= \exp\left(i\pi\mathcal{Z}^{AB}a_Aa_B + 2i\pi a_A\mathcal{Y}^A + 2i\pi a_A b^A\right)\ \theta(\mathcal{Y}+\mathcal{Z}a+b,\,\mathcal{Z})\\ &= \sum_{n^A\in\mathbb{Z}^M}\exp\left(i\pi\mathcal{Z}^{AB}(n_A+a_A)(n_B+a_B)+2i\pi(n_A+a_A)(\mathcal{Y}^A+b^A)\right)\end{aligned}$$

that also solve (3.11), result from the action of the HS symmetry (2.9) with $j_A = 2i\pi a_A$, $h^B = b^B$ and $\mu = \frac{i}{4\pi}$ on the theta function.

Consider the set of theta functions with equal characteristics $a^A = b_A = 0$ or $\frac{1}{2}$, $\forall A = 1, 2 \ldots M$,

$$\theta\left[\begin{smallmatrix}a\\a\end{smallmatrix}\right](\mathcal{Y},\,\mathcal{Z}) = \sum_{n^A\in\mathbb{Z}^M}\exp i\pi(\mathcal{Z}^{AB}(n_A+a_A)(n_B+a_B)+2(n_A+a_A)(\mathcal{Y}^A+a^A))\,, \tag{8.3}$$

which consists of 2^M independent functions. It is easy to see that

$$\theta\left[\begin{smallmatrix}a\\a\end{smallmatrix}\right](-\mathcal{Y},\,\mathcal{Z}) = (-1)^{\sum_{A=1}^M 2a_A}\ \theta\left[\begin{smallmatrix}a\\a\end{smallmatrix}\right](\mathcal{Y},\,\mathcal{Z})\,,$$

i.e., 2^{M-1} functions with an odd number of $a_A = \frac{1}{2}$ are odd in $\mathcal{Y}^A$ and 2^{M-1} functions with an even number of $a_A = \frac{1}{2}$ are even in $\mathcal{Y}^A$. In accordance with the normal relationship between spin and statistics, the odd functions describe half-integer spin massless fields while the even functions describe integer spin massless fields (in the former case the solution has to be multiplied by a Grassmann odd element). In particular, the theta function (8.2) is a member of this set with $a_A = 0$, i.e., it is bosonic.

The class of solutions $C^+(\mathcal{Y}|\mathcal{Z})$ (8.1) of the unfolded HS field equations in $\mathcal{M}_M$ is special in the sense that $C(Y|0)$ is not a regular function of Y as is usually assumed in the unfolded HS analysis, but becomes a distribution at real $\mathcal{Z}$.

Indeed, one can see that

$$\theta\left[\begin{smallmatrix}a\\a\end{smallmatrix}\right](\mathcal{Y},\,0) = \sum_{n^A\in\mathbb{Z}^M}(-1)^{\sum_{A=1}^M 2a_An^A}\delta^M(\mathcal{Y}^A+a^A-n^A)\,.$$

This formula means in particular, that $\theta\left[\begin{smallmatrix}a\\a\end{smallmatrix}\right](\mathcal{Y},\,\mathcal{Z})$ develops a singularity at $\mathcal{Z}\to 0$. As such, it is analogous to the $\mathcal{D}$-function of the massless field equations. In fact, it is the $\mathcal{D}$-function of HS field equations for solutions with appropriate (anti)periodicity conditions in $\mathcal{Y}$. In the limit in which the period of $\mathcal{Y}$ variables tends to infinity, i.e., the Fourier series in (8.2) is replaced by the Fourier integral, $\theta(\mathcal{Y},\,\mathcal{Z})$ becomes the $\mathcal{D}$-function as it is obvious from (6.5). The counterpart of the evaluation formula (6.23) for periodic in $\mathrm{Re}\,\mathcal{Y}$

function $C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$, conjugated to $C^+(\mathcal{Y}|\mathcal{Z})$ (8.1) is

$$C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = \int\limits_{[01)^M} d^M Y' \; \theta(Y' - \overline{\mathcal{Y}}|\mathcal{Z}' - \overline{\mathcal{Z}}) C^-(Y'|\mathcal{Z}') \tag{8.4}$$

with any $\mathcal{Z}'$. (The direct proof is analogous to that of Subsection 6.3.) The evolution formula for $C^+(\mathcal{Y},\mathcal{Z})$ is obtained with the help of $\theta^-(\overline{\mathcal{Y}},\overline{\mathcal{Z}}) = \overline{\theta(\mathcal{Y},\mathcal{Z})}$.

Also, let us note that the generalized Riemann identities for theta functions [22] are discrete analogues of the identity (7.11) for $\mathcal{D}^+$. For example, the generalized Riemann identity

$$\sum_{2a_A \in (\mathbb{Z}/2\mathbb{Z})^M} \theta\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right](2\mathcal{Y},\, 2\mathcal{Z})\theta\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right](2U,\, 2\mathcal{Z}) = \theta(\mathcal{Y}+U\,,\mathcal{Z})\theta(U-\mathcal{Y}\,,\mathcal{Z})$$

can be derived along the same lines as (7.12).

8.2 Theta functions and massless fields in Minkowski space

In the case of $M = 2$, eq. (8.3) gives solutions for massless scalar and spinor in three dimensions. In the case $M = 4$, it describes the superpositions of massless fields of all spins in four dimensions. Let us stress that direct identification of theta functions with solutions of $4d$ massless field equations turns out to be so simple just because, in the Sp(8) invariant framework, massless fields of all spins turn out to be involved. This is a manifestation of the general feature that linear and nonlinear field equations for massless fields of all spins are in a certain sense simpler than those for specific lower spins.

The reduction of the theta function solutions to a definite spin in the Minkowski subspace of $\mathcal{M}_M$ is more subtle. To this end, let us first of all make precise the relationship between Majorana indices $A, B, \ldots$ and two-component indices $\alpha, \beta \ldots$ and $\alpha', \beta' \ldots$. Let $A^A = (A^1, A^2, A^3, A^4)$. We set for two-component vectors $\tilde{A}$

$$\tilde{A}^\alpha = (A^1 + iA^3, A^2 + iA^4)\,, \qquad \tilde{A}^{\alpha'} = (A^1 - iA^3, A^2 - iA^4)\,.$$

The inverse relations are

$$A^1 = \frac{1}{2}(\tilde{A}^1 + \tilde{A}^{1'})\,, \quad A^2 = \frac{1}{2}(\tilde{A}^2 + \tilde{A}^{2'})\,, \quad A^3 = \frac{1}{2i}(\tilde{A}^1 - \tilde{A}^{1'})\,, \quad A^3 = \frac{1}{2i}(\tilde{A}^2 - \tilde{A}^{2'}).$$

Introducing

$$\nu_1 = n_1 - in_3\,, \quad \nu_2 = n_2 - in_4\,, \quad \nu_{1'} = n_1 + in_3\,, \quad \nu_{2'} = n_2 + in_4\,,$$

which describe points with integer coordinates on the complex plane, we observe that

$$n_A A^A = \nu_\alpha \tilde{A}^\alpha + \nu_{\alpha'} \tilde{A}^{\alpha'}\,.$$

Also we introduce the complex characteristics

$$\alpha_1 = a_1 - ia_3\,, \quad \alpha_2 = a_2 - ia_4\,, \quad \alpha_{1'} = a_1 + ia_3\,, \quad \alpha_{2'} = a_2 + ia_4\,.$$

As a result, the theta function with characteristics (8.3) can be rewritten as

$$\theta\left[{a \atop a}\right](\mathcal{Y},\,\mathcal{Z}) = \sum_{\mathrm{Re}\,\nu^A,\,\mathrm{Im}\,\nu^A\in\mathbb{Z}^4} \exp i\pi\big(\mathcal{Z}^{\alpha\beta}\mu_\alpha\mu_\beta + 2\mathcal{Z}^{\alpha\beta'}\mu_\alpha\mu_{\beta'} + \mathcal{Z}^{\alpha'\beta'}\mu_{\alpha'}\mu_{\beta'} + 2\mu_\alpha\tilde{\mathcal{Y}}^\alpha + 2\mu_{\alpha'}\tilde{\mathcal{Y}}^{\alpha'}\big)\,,$$

where

$$\mu_\alpha = \nu_\alpha + \alpha_\alpha\,, \qquad \mu_{\alpha'} = \nu_{\alpha'} + \alpha_{\alpha'}\,.$$

The solutions of spin s field equations in the complexified Minkowski space, that result from $\theta\left[{a \atop a}\right](\mathcal{Y},\,\mathcal{Z})$, are

$$C_{\alpha_1\ldots\alpha_{2s}}(\mathcal{Z}) = (2i\pi)^{2s} \sum_{\mathrm{Re}\,\nu^A,\,\mathrm{Im}\,\nu^A\in\,\mathbb{Z}^4} \mu_{\alpha_1}\ldots\mu_{\alpha_{2s}} \exp\left(2i\pi\mathcal{Z}^{\alpha\beta'}\mu_\alpha\mu_{\beta'}\right),$$

$$C_{\alpha'_1\ldots\alpha'_{2s}}(\mathcal{Z}) = (2i\pi)^{2s} \sum_{\mathrm{Re}\,\nu^A,\,\mathrm{Im}\,\nu^A\in\,\mathbb{Z}^4} \mu_{\alpha'_1}\ldots\mu_{\alpha'_{2s}} \exp\left(2i\pi\mathcal{Z}^{\alpha\beta'}\mu_\alpha\mu_{\beta'}\right).$$

9 Conclusion

The main technical result of this paper is the definition of the proper integration measure for the conserved charges of $4d$ massless fields in terms of an integral in the ten-dimensional matrix space $\mathcal{M}_4$. This allowed us not only to reproduce the known HS charges in Minkowski space starting from the ten-dimensional matrix space $\mathcal{M}_4$ but also to give the integral evolution formulae for massless fields via the $\mathcal{D}$-functions in $\mathcal{M}_4$. The precise integration prescription is given in terms of the Siegel upper half-space $\mathfrak{H}_M$ [21] of complex $M\times M$ symmetric matrices $\mathcal{Z}^{AB} = \mathcal{Z}^{BA}$ with positive definite imaginary parts. More generally, we observe that massless fields are most conveniently described in terms of the Siegel space with complex matrix coordinates. In this setup, positive- and negative-frequency solutions are described, respectively, as holomorphic and antiholomorphic functions in the Siegel upper half-space $\mathfrak{H}_M$. The systematic reformulation of the HS fields in the Siegel space leads to a number of surprising conclusions.

One is that the unfolded form of the classical massless field equations studied in this paper distinguishes between positive and negative frequencies, i.e., particles and antiparticles, the property that is usually delegated to the quantization prescription. We interpret this intriguing observation as the important indication that the unfolding procedure is able to describe quantization. It is worth to note that the unfolded equations themselves have a form of a multidimensional Schrodinger equation.

Another important observation is that Riemann theta functions provide a natural class of periodic solutions of the $4d$ massless field equations. This fact is a consequence of the Sp(8) and HS symmetries of the massless field equations. The setup of unfolded HS field equations is convenient for the analysis of properties of theta functions. For example, Riemann-type identities [22] can be derived by analysing solutions of the rank r equations (7.1) analogously to the analysis of massless field $\mathcal{D}$-functions in section 6.

Theta functions provide non-zero solutions of massless field equations that are invariant up to a phase factor $\sqrt[8]{1}$ under the transformations from the Igusa group $\Gamma_{1,2}\subset \mathrm{Sp}(2M,\mathbb{Z})$ [22]. As a result, theta functions may indeed play a distinguished role in the HS

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theory as highly symmetric nontrivial solutions because the observables constructed from such solutions, like, e.g., conserved currents, turn out to be invariant under $\Gamma_{1,2}$.

An intriguing possibility would be if such a solution can be identified with a nontrivial vacuum of the HS theory. Although any such a solution breaks down the Lorentz invariance to some its discrete subgroup, such a breakdown can be compatible with the observations if the scales of the periods of the vacuum solutions are large enough. In that case, the breakdown of the Lorentz invariance may have cosmological implications.

The fact of natural appearance of theta functions in the HS gauge theory is expected to shed light on a still mysterious relationship of HS gauge theories with String Theory and integrable systems.

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A Ring of solutions of massless equations

As observed in section 4, the space of solutions of the first-order unfolded current equation (4.2) forms a commutative associative algebra. A less trivial fact is that the space of solutions of the second-order unfolded equation (4.9) can also be endowed with the structure of associative commutative algebra as follows.

Let us introduce the following commutative and associative product $\circ$ on the space of fields $A(Y|X)$:

$$(A\circ B)\,(Y|X) = A(Y|X)\exp\left\{-2\mu\frac{\overleftarrow{\partial}}{\partial Y^A}X^{AB}\frac{\overrightarrow{\partial}}{\partial Y^B}\right\}B(Y|X). \tag{A.1}$$

The space of solutions of the unfolded system (1.3) is closed under the product $\circ$, i.e., given two solutions A, B of (1.3), $A\circ B$ is its new solution as is easy to see by straightforward substitution of (A.1) into (1.3).

The meaning of the product $\circ$ is quite simple. It corresponds to the usual product of the "initial data" $A(Y|0)$ and $B(Y|0)$ of the respective problems as follows from (A.1) at $X^{AB} = 0$. So, it is not too surprising that the space of solutions forms such an Abelian algebra. It is remarkable, however, that the product $\circ$ has the simple and constructive form (A.1) in $\mathcal{M}_M$. The integral version of the formula (A.1) with $\mu = 1$ reads as

$$(A\circ B)\,(Y|X) = \frac{det|X^{AB}|}{(2\pi)^M}\int dW\,dV\,A(Y+W|X)B(Y+V|X)\exp-\frac{1}{2}W^AV^BX_{AB}\,,$$

where the matrix X_{AB} is inverse to X^{AB}.

The generalization to rank r fields in $\mathcal{M}_M$ [10] is straightforward

$$\circ_{\otimes^r} = \exp\left\{-2\sum_{i=1}^{r} \frac{\overleftarrow{\partial}}{\partial Y_j^A} X^{AB} \frac{\overrightarrow{\partial}}{\partial Y_j^B}\right\}.$$

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